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Feasible joint posterior beliefs with binary signals ☆

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ABSTRACT

We extend a quantitative Aumann's agreement theorem in Arieli et al. (2021) from binary states to arbitrary finite states and provide a complete characterization of feasible joint posteriors beliefs when beliefs supports are binary. Using a network flow approach, we show that the martingale condition and the cut condition for a feasible flow are necessary and sufficient for feasible joint posterior beliefs. For symmetric supports, we show that the cut condition implies that the posteriors beliefs must be positively dependent. We also relate the cut condition to various characterization conditions in the literature.

1. Introduction

Aumann (1976) introduced a formal definition of common knowledge and used it to prove that two agents with the same priors about the world but different information cannot agree to disagree: If their posteriors for an event A are common knowledge, then these posteriors are equal. Aumann's Agreement Theorem effectively restricts the posterior beliefs of agents. Arieli et al. (2021) provided a linear characterization of feasible posterior beliefs and showed that these linear inequalities lead to a quantitative Aumann's Agreement Theorem. For arbitrary finite state space, a similar linear characterization of feasible joint posteriors as in Arieli et al. (2021) is a tough question. In this paper, we study this feasible joint posteriors problem with a finite state space for one of the simplest non-trivial cases, i.e., binary belief support for each agent. To obtain a characterization, we study this problem by a network flow approach. We show that together with the martingale condition, the cut condition for the existence of a feasible flow is necessary and sufficient for the feasibility.

Some recent literature has provided various characterizations of feasibility with many states. For many-agent problems, Morris (2020) provided a very general no-trade characterization of feasible joint posteriors by zero-value trades (with some measurability assumption). It can be shown that the cut condition corresponds to a particular choice of zero-value trades and hence is a lower-dimensional reduction of Morris's condition. For two-agent problems, Ziegler (2020) derived a class of necessary conditions for feasibility. His condition provides bounds on how dependent the beliefs can be across the agents in the spirit of Fréchet–Hoeffding bounds, by further tightening these classic bounds. Since these bounds concern cumulative distribution functions defined on beliefs, the spaces of beliefs need to be totally ordered. The cut condition, on the other hand, applies to the arbitrary beliefs including not totally ordered ones. Hence Ziegler's condition is implied by the cut condition and the martingale condition. For many agents, Levy et al. (2021) provided a class of necessary conditions including the martingale condition and a condition called the marginal expectation condition for any pair of agents. It can be shown that the equal marginal expectation condition is implied by a collection of agent-pairwise cut conditions and martingale conditions. Different from the first-order belief-based approach, for information design problems with multiple agents and binary signals, Arieli and Babichenko (2019), Bergemann and Morris (2019), and Taneva (2019) provided an alternative approach that relates information design problems to Bayes Correlated Equilibrium. Mathevet et al. (2020) provided an alternative characterization of feasibility implicitly by considering the entire belief hierarchy and showed that feasibility is equivalent to the consistency of the hierarchy.

2. Model

We consider two agents, 1 and 2, and a finite state space $\Omega = \{\omega_0, \omega_1, \ldots, \omega_m\}$, $m \geq 1$. The prior probability of state $\omega \in \Omega$

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is denoted by p_0^ω. The agents have common prior p_0 regarding the states. An information structure $I = (S, p)$ consists of a finite signal space $S = S_1 \times S_2$ and a distribution $p \in \Delta(\Omega \times S)$, with the marginal of p on Ω equal to p_0. Let (ω, s_1, s_2) be a realization. Then, each agent i observes the signal $s_i \in S_i$. Let $q_i(s_i) \in \Delta(\Omega)$ denote agent i's posterior belief about Ω after receiving the signal s_i. The posterior belief attributed to event $A \subset \Omega$ is given by

$$q_i^A(s_i) = p(A|s_i). \tag{1}$$

Denoted by p_S the marginal of p on the signal space S, and by p_S^i the marginal of p on the signal space S_i. We denote by ψ_I the joint distribution of posterior beliefs induced by I. That is, for each $(f_1, f_2) \in \Delta(\Omega) \times \Delta(\Omega)$, define

$$\psi_I(f_1, f_2) = p(q_1 = f_1, q_2 = f_2). \tag{2}$$

For each agent i, let $F_i \subset \Delta(\Omega)$ be a set of posterior beliefs. Denote $F = F_1 \times F_2$. We say F is binary if for each agent i, F_i contains two elements. For a probability measure $\psi \in \Delta(F)$ and for agent i, we denote by ψ_i the marginal distribution of the posterior of agent i. We can start with an arbitrary (F, ψ) and ask whether it can be generated by some information structure.

Definition 1. $\psi \in \Delta(F)$ is p_0-feasible if there exists some information structure I with prior p_0 such that $\psi = \psi_I$.

3. Aumann's theorem revisit

We first present the following quantitative Aumann's Agreement Theorem with finitely many states, which follows from Theorem 1 in Arieli et al. (2021) for binary states by applying additivity of probabilities.

Theorem 1. *Let (S, p) be an information structure. Let $A \subset \Omega$ be an event. (a) The agents' average posterior beliefs are approximately equal when they are approximately common knowledge: for all $E_1 \subseteq S_1$, $E_2 \subseteq S_2$,*

$$-p_S(E_1^c \times E_2) \le \mathbb{E}(q_1^A \mathbb{1}_{s_1 \in E_1}) - \mathbb{E}(q_2^A \mathbb{1}_{s_2 \in E_2}) \le p_S(E_1 \times E_2^c). \tag{3}$$

(b) (Aumann's Agreement Theorem) For each i, let E_i be the set of signals that i has posterior r_i for A. If $q_1^A = r_1$ and $q_2^A = r_2$ is common knowledge, then $r_1 = r_2$.

Proof (b)**.** is similar to Arieli et al. (2021) and we show (a) holds. Fix $A \subset \Omega$ and consider a new binary state space $\tilde{\Omega} = \{\tilde{\omega}_0, \tilde{\omega}_1\}$ where for each probability measure p_0 in the original problem, define a corresponding one for the new problem by $\tilde{p}_0^{\tilde{\omega}_1} = \sum_{\omega \in A} p_0^\omega$. Then for the new problem Theorem 1 of Arieli et al. (2021) applies. ■

4. Characterization

Theorem 1 immediately implies the following necessary condition for feasibility.

Corollary 1. *Let $\psi \in \Delta(F)$ be a joint distribution of posterior beliefs. If ψ is p_0-feasible, then*

$$-\psi(C_1^c \times C_2) \le \sum_{f_1 \in C_1}\Big(\sum_{\omega \in A} f_1^\omega\Big)\psi_1(f_1) - \sum_{f_2 \in C_2}\Big(\sum_{\omega \in A} f_2^\omega\Big)\psi_2(f_2) \le \psi(C_1 \times C_2^c), \tag{4}$$

for all $A \subset \Omega$ and all $C_1 \subseteq F_1, C_2 \subseteq F_2$.

Proof. Suppose ψ is p_0-feasible with information structure (S, p), then for each $f_i \in F_i$, there exists $s_i \in S_i$ such that $q_i(s_i) = f_i$. Then for each $C_i \subseteq F_i$, define $E_i = \{s_i \in S_i : q_i(s_i) \in C_i\}$ in Theorem 1, we obtain condition (4). ■

For binary belief supports, we obtain that condition (4) is necessary and sufficient for feasibility.[1]

Theorem 2. *Let $\psi \in \Delta(F)$ be a joint distribution of posterior beliefs and F is binary. ψ is p_0-feasible for some p_0 if and only if (4) holds.*

4.1. Proof of Theorem 2

Corollary 1 has shown that condition (4) is necessary for feasibility. We next show that for binary belief supports, this condition is also sufficient for feasibility. In the following of the section, for any $\psi \in \Delta(F)$ we define $h_i : \Omega \times F_i \to \mathbb{R}$ by

$$h_i^\omega(f_i) := f_i^\omega \psi_i(f_i), \quad \text{for all } i = 1, 2.$$

First note that condition (4) can be decomposed into the following two conditions:

(1) *The martingale condition* (Blackwell, 1951; Aumann and Maschler, 1995; Kamenica and Gentzkow, 2011): for all $\omega \in \Omega$, $i = 1, 2$,

$$\sum_{f_i \in F_i} h_i^\omega(f_i) = p_0^\omega. \tag{5}$$

(2)*The cut condition*: for all $A \subseteq \Omega$, $C_1 \subseteq F_1$, $C_2 \subseteq F_2$,

$$\sum_{\omega \in A}\Big[\sum_{f_1 \in C_1} h_1^\omega(f_1) - \sum_{f_2 \in C_2} h_2^\omega(f_2)\Big] \le \psi(C_1 \times C_2^c). \tag{6}$$

As we will show below, condition (6) is associated with the cut sets in a network flow problem and we call this condition *the cut condition*.

To obtain the characterization, we show that the feasible joint posteriors problem can be transformed into a network flow problem and a maximum-flow minimum-cut theorem can be invoked to this problem. Below we assume agents' belief supports are binary, i.e., $F_i = \{f_{i1}, f_{i2}\}$.

Lemma 1. *$\psi \in \Delta(F)$ is p_0-feasible if and only if ψ satisfies the martingale condition (5) and the following linear system has a feasible solution $(Y^\omega(f_2))_{\omega \in \Omega, f_2 \in F_2}$:*

$$\sum_{f_2 \in F_2} Y^\omega(f_2) = h_1^\omega(f_{11}), \text{for all } \omega \in \Omega, \tag{7}$$

$$\sum_{\omega \in \Omega} Y^\omega(f_2) = \psi(f_{11}, f_2), \text{for all } f_2 \in F_2, \tag{8}$$

$$0 \le Y^\omega(f_2) \le h_2^\omega(f_2), \text{for all } \omega \in \Omega, f_2 \in F_2. \tag{9}$$

Proof. Suppose ψ is p_0-feasible and let (S, p) be one corresponding information structure. For all ω, f_1, f_2, define $p(\omega, f_1, f_2) = p(\omega, q_1 = f_1, q_2 = f_2)$. Then from p_0-feasibility of ψ,

$$p(\omega, f_{11}, f_2) + p(\omega, f_{12}, f_2) = h_2^\omega(f_2), \tag{10}$$

for all $\omega \in \Omega, f_2 \in F_2$, it implies that we can take $p(\omega, f_{12}, f_2)$ as a slack variable. Define $Y^\omega(f_2) = p(\omega, f_{11}, f_2)$ for all $\omega \in \Omega, f_2 \in F_2$. Then $(Y^\omega(f_2))_{\omega \in \Omega, f_2 \in F_2}$ is a feasible solution to (7)–(9). Conversely, suppose $(Y^\omega(f_2))_{\omega \in \Omega, f_2 \in F_2}$ is a feasible solution to (7)–(9). Define $S_i = F_i$ and $p(\omega, f_{11}, f_2) = Y^\omega(f_2)$ and $p(\omega, f_{12}, f_2) = h_2^\omega(f_2) - Y^\omega(f_2)$. Define $p_0^\omega = \sum_{f_2 \in F_2} Y^\omega(f_2)$. Then (S, p) is the required information structure. Hence ψ is p_0-feasible. ■

[1] A direct revelation argument implies that for any information structure (ω, s_1, s_2), there is an equivalent structure (ω, q_1, q_2) in which agent i observes the posterior belief induced by s. So for binary belief supports, it is without loss to consider binary signal spaces. We will use binary belief supports and binary signals interchangeably.

We next apply a network flow analysis to the linear system (7)–(9). Define a network flow problem (V, E, c) as follows: the node set V contains a source v_0, a bipartite node set $\Omega \cup F_2$, and a sink v_1, and the arc set E consists of arcs from v_0 to each ω, from each ω to each f_2, and from each f_2 to v_1. Define the capacity function $c : E \to \mathbb{R}$ by $c(v_0, \omega) = h_1^{\omega}(f_{11})$, $c(\omega, f_2) = h_2^{\omega}(f_2)$, and $c(f_2, v_1) = \psi(f_{11}, f_2)$. For any $v \in V$, denote $\delta^{in}(v)$ as the set of the arcs entering v and $\delta^{out}(v)$ as the set of arcs leaving v. We say $x : E \to \mathbb{R}$ is a feasible flow if x satisfies the following two conditions:

(1) flow conservation constraints at non-terminal nodes:

$$\sum_{a\in\delta^{out}(v)} x(a) = \sum_{a\in\delta^{in}(v)} x(a), \text{ for all } v \notin \{v_0, v_1\}, \tag{11}$$

(2) capacity constraints on arcs:

$$0 \le x(a) \le c(a), \text{ for all } a \in E. \tag{12}$$

Denote $val(x)$ the value of flow x by $\sum_{a\in\delta^{out}(v_0)} x(a)$. A set G of arcs is a cut if $G = \delta^{out}(U)$ for some subset $U \subset V$ with $v_0 \in U$ and $v_1 \notin U$. Denote $cap(\delta^{out}(U))$ the total capacity from the arcs $\delta^{out}(U)$. A maximum flow problem is to find a feasible flow with the maximum value. The following lemma establishes an equivalence between the joint posterior feasibility problem and the maximum flow problem.

Lemma 2. *The linear system (7)–(9) has a feasible solution if and only if the network flow problem (V, E, c) has a maximum flow x^* satisfying*

$$val(x^*) = \sum_{\omega\in\Omega} h_1^{\omega}(f_{11}) = \sum_{f_2\in F_2} \psi(f_{11}, f_2). \tag{13}$$

Proof. Suppose (7)–(9) has a feasible solution Y. Then Y satisfies (11)–(12) and is a feasible flow. Moreover, Y is a maximum flow as $val(x) \le \sum_{\omega\in\Omega} h_1^{\omega}(f_{11})$ for all feasible flows x. Conversely, suppose (V, E, c) has a maximum flow x^* satisfying (13). Then all capacity constraints at arcs (v_0, ω) and (f_2, v_1) must hold with equality. x^* is a feasible solution to (7)–(9). ■

From Lemmas 1 and 2, we have transformed the original problem into the maximum flow problem (V, E, c). By a max-flow min-cut theorem (e.g. Schrijver, 1986), for every cut $U \subset V$,

$$\max_x val(x) \le cap(\delta^{out}(U)). \tag{14}$$

Combine (13) and (14), we get

$$\sum_{\omega\in\Omega} h_1^{\omega}(f_{11}) = \sum_{f_2\in F_2} \psi(f_{11}, f_2) \le \sum_{\omega\notin U} h_1^{\omega}(f_{11}) + \sum_{\omega\in U, f_2\notin U} h_2^{\omega}(f_2) + \sum_{f_2\in U} \psi(f_{11}, f_2). \tag{15}$$

Notice that any U is of the form $\{v_0\} \cup A \cup C_2$ for some $A \subseteq \Omega$, $C_2 \subseteq F_2$. There are two cases either $\omega_0 \notin A$ or $\omega_0 \in A$. In each case, some algebraic manipulation shows that (15) reduces to the cut condition (6). This completes the proof.

5. The cut condition and correlation

In this section, we show that the characterization condition in Theorem 2 imposes strong restrictions on the bounds and possible forms of dependence structures. Roughly, the cut condition requires that the posterior beliefs are not too negatively dependent.

Consider a class of joint distributions of beliefs with arbitrary finite symmetric belief supports. With symmetric supports, we introduce the following intuitive notion for positive dependence. It requires that the joint distribution of beliefs puts more "weights" on the diagonal points than the average of off-diagonal points, exhibiting a form of positive dependence.

Definition 2. Suppose F is symmetric ($F_1 = F_2$) and let $\psi \in \Delta(F)$. We say ψ is average affiliated, if for all $f \in F_1$,

$$\psi(f, f) \ge \frac{1}{|F_1| - 1} \max\{\sum_{f'\neq f} \psi(f, f'), \sum_{f'\neq f} \psi(f', f)\}. \tag{16}$$

Below we discuss a class of problems with $m \ge 2$. We say a joint distribution ψ is a star, if $F_1 = F_2 = \{f(1), \dots, f(m)\}$ where for each $f(j)$, $j = 1, \dots, m$, we have

$$f^{\omega}(j) = \begin{cases} \delta & \text{if } \omega = \omega_0, \\ 1-\delta & \text{if } \omega = \omega_j, \\ 0 & \text{if } \omega \neq \omega_0, \omega_j. \end{cases}$$

From the martingale condition, we have $\delta = p_0^{\omega_0}$. By construction, there is a one-to-one correspondence between each $f(j) \in F_1$ and $\omega_j \in \overline{\Omega}$, i.e., the agent knows that the state is either ω_j or ω_0. The support assumption further requires that each agent's posterior for state ω_0 remains the prior and the posterior for any other state shifts all mass to some ω_j. When δ approaches zero, the signals reveal almost perfect information.

We now apply the cut condition in Theorem 2 to a star ψ. Consider the following class of testing sets $(C_1, C_2) = (C, C^c)$ for some $C \subseteq \overline{\Omega}$ (note that we can denote F_1 by $\overline{\Omega}$). The cut inequalities require that for all $C \subseteq \overline{\Omega}$ and $A \subseteq \overline{\Omega}$, we have

$$-\psi(C^c\times C^c) \le (1-\delta)[\psi((C\cap A)\times\overline{\Omega}) - \psi(\overline{\Omega}\times(C^c\cap A))] \le \psi(C\times C). \tag{17}$$

It implies that the tightest upper bound is obtained at $A = C$ and the lower bound is obtained at $A = C^c$, and the condition can be simplified as

$$\psi(C \times C) \ge \frac{1-\delta}{\delta} \max\{\psi(C \times C^c), \psi(C^c \times C)\}. \tag{18}$$

Intuitively, $\psi(C \times C^c)$ denotes the probability that two agents have opposite beliefs on the possible states. $\psi(C \times C)$ denotes the probability that agents' beliefs are aligned. The cut inequalities then say that agents' beliefs cannot be too opposite. Moreover, take $C = \{j\}$, $j = 1, \dots, m$, we obtain that for $p_0^{\omega_0} \le 1 - 1/m$, the cut inequalities imply average affiliation.

Proposition 1. *Suppose $p_0^{\omega_0} \le 1 - 1/m$ and $\psi \in \Delta(F)$ is a star. If ψ is p_0-feasible, then ψ is average affiliated.*

Data availability

No data was used for the research described in the article

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