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A novel time-power based grey model for nonlinear time series forecasting

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ABSTRACT

To deal with various nonlinear issues in real applications, a novel time-power based grey model is put forward. However, in the original form of this model, the time-power parameter α normally equals to an integer, and then the analytical expression of the time response function will be obtained. Otherwise, if the parameter α equals to a non-integer, one cannot obtain the concrete time response function for future estimations. This situation may significantly restrict the applications of this grey model. To address such drawbacks, an optimized version is designed in this work. In the proposed model, a simplified solution to the differential equation is derived by using the definite integral technique. Furthermore, for improving accuracy, the time-power parameter α is optimized by utilizing the Particle Swarm Optimization algorithm based on the model parameter packages. Subsequently, the efficacy and practicality of this simplified function have been verified by numerical simulations and experimental studies. Moreover, the method of probability density prediction is employed for verifying the reliability and stability of the proposed model for the first time when predicting the settlement of the soft-clay subgrade on an expressway. The demonstration cases illustrate that the quantitative improvements over forecasts of the proposed model are even more pronounced with a level accuracy of 2.29% and 1.19% MAPE values in the fitted and predicted periods, respectively, which can significantly increase the predicting accuracy by more than 10% with respect to the other benchmarks. Therefore, the new proposed model not only has greater application fields and prospects but also achieves higher and more reliable predicting accuracy with the optimal α under the support of the Particle Swarm Optimization algorithm, compared with the competing models.

1. Introduction

The main purposes of modeling and forecasting time series are to reveal the inherent law of a system and estimate its future trends, which plays an essential role in decision-making (Zeng et al., 2019; Ma et al., 2020). Currently, modeling nonlinear issues has emerged as a hot topic and many methods have been proposed, such as statistical methods, neural network algorithms, and grey prediction models (Xiao et al., 2017; Wang and Hao, 2016; Xiong et al., 2019; Zhou et al., 2021). Among these models, the grey prediction models have attracted enormous attention because of their excellent ability for dealing with system projections characterized by uncertainty and incomplete information (Deng, 1982; Wang et al., 2017; Liu and Zhang, 2019; Mao et al., 2020). Accordingly, grey prediction models have been widely used in the economic, industrial, natural systems, and other fields (Wang, 2013; Ma and Liu, 2018; Ding, 2019; Chen et al., 2020).

As the most important branch of the grey models, the grey model having one variable and one order, abbreviated as $GM(1,1)$, is widely used by many researchers (Zheng et al., 2018; Xiao and Duan, 2020). Moreover, a range of studies have optimized this model from diverse perspectives, which generally include the following aspects:

(1) Optimization of the background value and grey derivative. Xu et al. (2017) modified the background and used this optimized model for projections of China's electricity demand. Li et al. (2011) used the cubic spline function to optimize the background value and grey derivative of the $GM(1,1)$ model. The optimized model was applied to the short-term power load modeling and obtained good results. Lin et al. (2012) developed an improved artificial fish swarm algorithm by minimizing the average relative errors to identify the parameters. Additionally, to further improve a model's accuracy, a rolling mechanism is frequently used to build an optimized model. Evans (2014) gave

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another estimation method of the $GM(1,1)$ and Verhulst model parameters by using determinable coefficients. Applying the improved Verhulst model to analyze the steel utilization efficiency in the UK, Evans found that the new estimation method possessed accurate and reliable multi-step prediction results. Although many works about optimizing the background value of the *GM(1,1)* model have been performed, they cannot change the situation that this grey model is still suitable to model the homogeneous time sequences, which might strongly restrict the application fields (Bezuglov and Comert, 2016; Ding et al., 2021a).

(2) Optimization of model parameters and time response function. Zhao et al. (2012) incorporated the rolling mechanism into the hybrid grey model with a differential evolution algorithm and found that the novel model could significantly improve the forecasting precision in comparison with benchmark models. Ding et al. (2018) introduced the dynamic weighting coefficients. Simultaneously, these weight parameters were solved through the Artificial Intelligence Optimization Algorithm. Nevertheless, it still existed a problem that the sum of the cumulative sequence weight coefficient did not equal to one, which meant that the normalization of the model needed to be improved. Making the sum of error square minimum, Xie and Liu (2009) improved the initial conditions of the $DGM(1,1)$ model, and the results showed that the optimization model can effectively predict the society and economic system. Xu et al. (2019) combined a new grey model with a buffered rolling method to further improve forecasting performance. This proposed model can capture the essential characteristics of a developing trend in comparison with conventional grey models. Moreover, empirical results also demonstrate its superior performance over other competitors. However, these improved models have little advantages in describing nonlinear time sequences because they still possess linear model structures (Ding and Li, 2021).

(3) Construction of new grey models. As the research methods are more and more abundant and the system diversity is gradually recognized, many grey prediction models have been designed for the new characteristics of the system. Wu et al. (2019a,b) designed a seasonal $GM(1,1)$ model to describe the seasonal features of the pollutants, such as NO_2, $PM_{2.5}$, and CO. Zhu et al. (2020) proposed a self-adaptive grey fractional weighted model by introducing the fractional weighted coefficients to design a novel initial condition. Furthermore, this new model is employed for predicting Jiangsu's electricity demand in comparison with a range of benchmark models. Zeng et al. (2020) designed a new grey Verhulst model by incorporating a non-homogeneous exponential function, which enables the novel model to solve various nonlinear problems with stronger adaptability. The extended grey models mentioned above can deal time series with various kinds of data characteristics, such as seasonality, volatility, and complexity. However, the way to modify the *GM* models is expected to make the optimized models more complicated, which may bring many difficulties in estimating parameters (Comert et al., 2021).

(4) Extensions of $GM(1,1)$ or modeling nonlinear sequences. For the reason that a lot of non-linear time series exist in the real world, proposing an effective forecasting technique is necessary for addressing such predicting issues. Cui et al. (2013) proposed a new grey model, named $NGM(1,1,k)$, according to the non-homogeneous exponential law of real data. The experimental results showed that $NGM(1,1,k)$ had good forecasting accuracy. For further improving its performance, Liu et al. (2016), Tong et al. (2017), and Ding et al. (2017) modified the background values of this model from different perspectives. Besides, based on the idea of "in-between", Wu et al. (2015, 2014) proposed a fractional-order accumulating generation operator for grey models, which breaks the restrictions that the order of the grey accumulating generation can only be an integer. Furthermore, to adapt to various features of a system sequence, such as non-homogeneous and homogeneous exponential sequences, and dynamic sequences featured by dynamic changing growth rate, Qian et al. (2012) initially proposed the grey model having one variable, one order and α time power, namely $GM(1,1,t^{\alpha})$, and analyzed the properties of the model with different values of the parameter α. In general, diverse values of α enable the grey model to simulate and predict different categories of time sequences, such as the homogenous exponential sequence (Wu et al., 2014; Zhou and Ding, 2021), non-homogenous exponential sequence (Cui et al., 2013; Ding et al., 2017), and a series featured by S shape (Qian et al., 2012; Ding et al., 2021b). Thus, due to its enough adaptive capability, this model has been widely used to solve various kinds of problems. Subsequently, for further revealing the inherent features of the $GM(1,1,t^{\alpha})$ model, many researchers conducted studies in different ways. Cui et al. (2016) measured the morbidity of this model by using the technique named the spectrum condition number of the matrix. He found that there exists no severe morbidity when calibrating the $GM(1,1,t^{\alpha})$ model. Wu et al. (2019a,b) modified the modeling structure of the conventional $GM(1,1,t^{\alpha})$ model, and determine the optimal initial point by minimizing the target function. Then, the efficacy and applicability were demonstrated in the experiments. Guo et al. (2014) combined the self-memory mechanism with the conventional $GM(1,1,t^{\alpha})$ model for improving forecasting precision. Guo et al. (2015) proposed a non-equidistance $GM(1,1,t^{\alpha})$ model based on the unequal interval sequences over time. She also discussed the relationship between the parameter α and the model's curve, power exponent, and the development coefficients.

In general, the $GM(1,1,t^{\alpha})$ model is able to address non-linear issues with good performance. However, it still has certain inherent shortcomings that seriously slash the adaptability and applicability of this model (Zeng et al., 2016; Liu and Xie, 2019). By analyzing the mechanism of the $GM(1,1,t^{\alpha})$ model, some findings can be obtained as follows: when α is an integer, one can get the analytic expression of the time response function. Otherwise, because α is not an integer and the integrand does not exist the original function, the time response function cannot be directly expressed by using the analytic formula (explained in Section 2). Specifically, Qian et al. (2012) only provided the time response functions used to generate forecasts when α equals 0, 1, and 2. Nevertheless, the authors cannot present the precise solutions to the differential equations when α equals other values, such as larger integer and non-integer. Because it is hard to solve the differential equation in Eq. (3). Based on the previous findings, one can conclude that the conventional $GM(1,1,t^{\alpha})$ model has limitations in producing accurate projections when the parameter α is a non-integer. This situation may restrict the application fields of the $GM(1,1,t^{\alpha})$ model. To address such drawbacks mentioned above, several contributions can be made as follows:

(1) The advantages and disadvantages of the conventional $GM(1,1,t^{\alpha})$ model are discussed, which provided solid foundation for understanding its modeling mechanism and facilitating building an optimized one. Moreover, graphic and visual effects of the several representative values of the parameter α are presented to illustrate the model's vast application fields.

(2) A simplified solution to the differential equation is derived by using the definite integral technique to overcome the inherent shortcomings in the $GM(1,1,t^{\alpha})$ model, which composes the core component of the optimized $GM(1,1,t^{\alpha})$ model, short for $OGM(1,1,t^{\alpha})$. Such improvements enable the new model to effectively address time series forecasting issues with various characteristics, such as nonlinearity and volatility, by adaptatively adjusting the parameter α without integer restrictions.

(3) For elaborating on the process of calibrating parameters, the parameter packages are initially introduced in this paper. This work is beneficial for novice readers to grasp the mechanism of parameter estimations in the new model. Furthermore, based on these parameter packages, the optimal values of the parameter α can be determined by utilizing a certain intelligent algorithm, such as the Particle Swarm Optimization algorithm (PSO) (Kennedy, 2010; Ding et al., 2021b).

(4) An experimental study on forecasting the settlement of soft-clay subgrade for an expressway is conducted for verifying the efficacy and adaptability of the proposed model. A range of competing models is

selected to compare with the proposed model, such as $GM(1,1,t^1)$, $GM(1,1,t^2)$, the grey power model ($GPM(1,1)$) (Wang et al., 2009), the nonlinear discrete grey model ($NDGM(1,1)$) (Zhang and Liu, 2010), Autoregressive Integrated Moving Average model ($ARIMA$), and Back Propagation Neural Network ($BPNN$). Moreover, a method of probability density prediction provides an alternative perspective for further illustrating the reliability and stability of this proposed model, besides the widely used indicators: Absolute Percentage Error (APE), Mean Absolute Percentage Error ($MAPE$), and Root Mean Squared Error ($RMSE$).

The structure of this paper is designed below: Section 2 is dedicated to the introduction of the modeling mechanism of the conventional $GM(1,1,t^\alpha)$ model. Moreover, the classical model's advantages and disadvantages is discussed here. Based on the findings in Section 2, the newly proposed $OGM(1,1,t^\alpha)$ model is designed, and the details of the parameter estimation and differential equation solution are provided in Section 3. Besides, two experiments and their analysis are given in Section 4. Finally, the conclusions of this paper are presented in Section 5.

2. The conventional GM(1,1,t$^\alpha$) model

The conventional $GM(1,1,t^\alpha)$ model is originally proposed by Qian et al. (2012), whose purpose is to forecast a system sequence characterized by a dynamic changing growth rate, normally S-sharped curve. This curve is usually divided into three phases: uniform phase with low speed, acceleration phase with mid-high speed, and stable phase with low speed. For this situation, the traditional $GM(1,1)$ model that is good at modeling homogenous sequences cannot perform well. Thus, the $GM(1,1,t^\alpha)$ model is necessary for further expanding the application areas of the grey prediction theory. Subsequently, the details of this model will be provided as follows.

Definition 1 (*Qian et al., 2012*). Set $\mathbf{X^{(0)}} = (x^{(0)}(1), x^{(0)}(2), \dots, x^{(0)}(n))$, $n \geq 4$ is a non-negative sequence, where n is the size of a sequence and $x^{(0)}(k)$ represents the *kth* actual observation. The first order accumulating generation sequence of $\mathbf{X^{(0)}}$ is $\mathbf{X}^{(1)} = (x^{(1)}(1), x^{(1)}(2), \dots, x^{(1)}(n))$, $n \geq 4$, where $x^{(1)}(k) = \sum_{i=1}^{k} x^{(0)}(i)$, $k = 1, 2, \dots, n$. By using the first order Accumulation Generation Operator (*1-AGO*) (Deng, 1982; Liu and Forrest, 2010), the randomness and noise in the original domain can be significantly reduced and the inner patterns will be enhanced, which can increase the forecasting precision.

Subsequently, $\mathbf{Z^{(1)}} = (z^{(1)}(1), z^{(1)}(2), \dots, z^{(1)}(n))$ is the sequence of the mean generation of $X^{(1)}$, where $z^{(1)}(1) = x^{(1)}(1)$, $z^{(1)}(k) = \frac{1}{2}(x^{(1)}(k) + x^{(1)}(k-1))$, $k = 2, 3, \dots, n$. $\mathbf{Z^{(1)}}$ is called the background values, which works as a bridge connecting the difference and differential equations (noted in Eqs. (1) and (2), respectively)

Definition 2 (*Qian et al., 2012*). $X^{(0)}$, $X^{(1)}$, and $Z^{(1)}$ are given in Definition 1, the equation

$$x^{(0)}(k) + az^{(1)}(k) = bk^\alpha + c \tag{1}$$

is called the basic form of the $GM(1,1,t^\alpha)$ model, where a refers to the development coefficient, b represents a nonlinear coefficient, and c is the grey action quantity term, respectively. Then, the equation

$$\frac{dx^{(1)}}{dt} + ax^{(1)}(k) = bt^\alpha + c \tag{2}$$

is named as the whitening equation of the $GM(1,1,t^\alpha)$ model.

The difference equation in Eq. (1) is functionally used to estimate the parameters a and b with the known α. Subsequently, by substituting the estimated parameters a and b into Eq. (2) and solving the whitening differential equation, one can obtain the time response function for producing predictions. Detailed process to parameter estimation will be illustrated in Theorem 1.

Theorem 1 (*Qian et al., 2012*). *$\mathbf{X^{(0)}}$, $\mathbf{X^{(1)}}$, and $\mathbf{Z^{(1)}}$ are defined in Definition 1, one can obtain the estimated values of a, b, and c, namely $\hat{\mathbf{r}} = [a, b, c]^T = (\mathbf{B}^T\mathbf{B})^{-1}\mathbf{B}^T\mathbf{Y}$, where*

$$\mathbf{B} = \begin{bmatrix} -z^{(1)}(2) & 2^\alpha & 1 \\ -z^{(1)}(3) & 3^\alpha & 1 \\ \vdots & \vdots & \vdots \\ -z^{(1)}(n) & n^\alpha & 1 \end{bmatrix}, \mathbf{Y} = \begin{bmatrix} x^{(0)}(2) \\ x^{(0)}(3) \\ \vdots \\ x^{(0)}(n) \end{bmatrix}.$$

Proof. Substituting the values of k over different time into Eq. (1), one will have

$$\begin{aligned} x^{(0)}(2) + az^{(1)}(2) &= 2^\alpha b + c \\ x^{(0)}(3) + az^{(1)}(3) &= 3^\alpha b + c \\ &\vdots \\ x^{(0)}(n) + az^{(1)}(n) &= n^\alpha b + c \end{aligned}$$

Then, solving the matrix $\mathbf{Y} = \mathbf{B}\hat{\mathbf{r}}$ by utilizing the ordinary least square method, the estimated values of the parameter can be obtained $\hat{\mathbf{r}} = [a, b, c]^T = (\mathbf{B}^T\mathbf{B})^{-1}\mathbf{B}^T\mathbf{Y}$.

Theorem 2 (*Qian et al., 2012*). *After estimating the values of the parameters a, b, and c, the time response function for solving the whitening equation in Eq. (2) is presented as*

$$x^{(1)}(t) = be^{-at}\int t^\alpha e^{at}dt + C \tag{3}$$

where a and b can be estimated by the ordinary least squares method. The C is a constant that depends on the initial value $x^{(1)}(1)$.

Proof. Using the solution of the first-order linear nonhomogeneous ordinary differential equation, Eq. (3) can be obtained.

As introduced by Qian et al. (2012), the $GM(1,1,t^\alpha)$ model has certain advantages, such as unifying a family of univariate grey models accompanying with different values of α and modeling various time sequences with different characteristics. To be specific, when the parameter $\alpha = 0$, $GM(1,1,t^\alpha)$ yields to the $GM(1,1)$ model (Zeng et al., 2020), which is normally used to model the homogenous series. When the parameter $\alpha = 1$, $GM(1,1,t^\alpha)$ is equivalent to the $GM(1,1,k)$ model (Guo et al., 2014; He et al., 2019; Kennedy, 2010), which is usually utilized to model the non-homogenous sequences. When the parameter α is assigned with other integer values, the $GM(1,1,t^\alpha)$ model can model certain sequences having various characteristics. In general, the $GM(1,1,t^\alpha)$ model is capable of representing the most popular homogeneous and non-homogeneous grey models, and it can also induce diverse other new models. As can be seen in Fig. 1, when the parameter α is assigned with different values, the $GM(1,1,t^\alpha)$ model can adjust to many modeling sequences having a diverse growing rate. It means that various time series can be described by adopting different α values, showing strong adaptability and broad practicality.

Although the $GM(1,1,t^\alpha)$ model has strong adaptability with various values of α, it still gets some disadvantages. Specifically, the parameter α normally equals an integer (usually $\alpha = 0, 1$ *or* 2) in previous studies because it is easy to solve the differential function in Eq. (2). This limitation may restrict the expanded applications of grey models. Moreover, accurate solutions to the differential equation in Eq. (2) are not provided when the parameter α equals other values except for zero, one, and two. This situation may bring challenges for the $GM(1,1,t^\alpha)$ model to model system sequences with different features in the real world, thereby restricting its popularization and application. Thus, to overcome these limitations, proposing a novel method to solve to differential functions accurately is imperative to enhance the generality and practicality of the conventional $GM(1,1,t^\alpha)$ model.

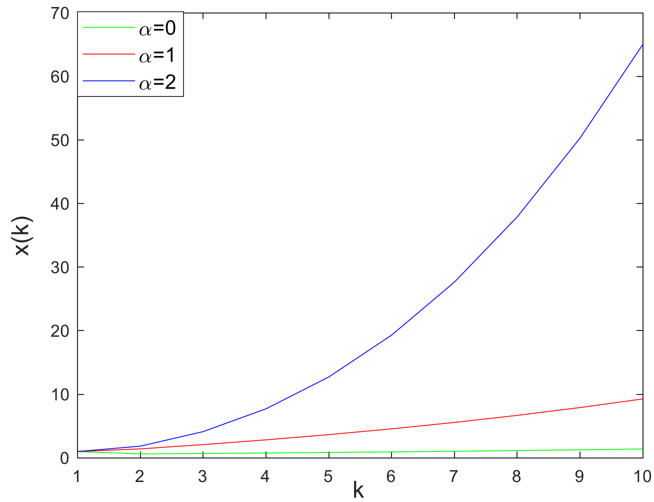

Fig. 1. Visual expressions of the $GM(1,1,t^{\alpha})$ model when α equals 0, 1, and 2.

3. The proposed OGM(1,1,t^{α}) model

To address the above disadvantages of the conventional $GM(1,1,t^{\alpha})$ model, a new optimized grey model, namely $OGM(1,1,t^{\alpha})$ model, is designed. The following subsections are dedicated to elaborating on the mechanism of the newly proposed model.

3.1. The simplified time response function for the proposed model

As introduced in Definitions 1 and 2, and Theorem 1, the basic concepts of the proposed model are similar to the conventional $GM(1,1,t^{\alpha})$ model. Subsequently, one will concentrate on the solution to the differential equations in Eq. (2).

Theorem 3. *After estimating the values of the parameters a, b, and c according to Theorem 1, the simplified time response function for solving the whitening equation in Eq. (2) is obtained as:*

$$x^{(1)}(t) = e^{-at}\int_{1}^{t} b t^{\alpha} e^{a\tau} d\tau + \frac{c}{a} - \frac{c}{a}e^{-a(t-1)} + x^{(1)}(1)\, e^{-a(t-1)} \tag{4}$$

Then, the corresponding discrete version of the time response function is

$$x^{(1)}(k) = e^{-at}\int_{1}^{k} b\tau^{\alpha} e^{a\tau} d\tau + \frac{c}{a} - \frac{c}{a}e^{-a(k-1)} + x^{(1)}(1)\, e^{-a(k-1)}, \quad k = 1,2,\ldots \tag{5}$$

Moreover, the forecasted values in the original domain can be obtained by using the one-order Inverse Accumulating Generating Operation (abbreviated as $1-IAGO$) (Deng, 1982; Liu and Forrest, 2010) as follows.

$$\hat{x}^{(0)}(k+1) = \hat{x}^{(1)}(k+1) - \hat{x}^{(1)}(k), \quad k = 1,2,\ldots \tag{6}$$

Proof. By solving the differential equation in Eq. (2), one can obtain

$$x^{(1)}(t) = e^{-\int a dt}\left[C + \int (bt^{\alpha} + c)\, e^{\int a dt} dt\right] \tag{7}$$

where a, b, and c can be estimated by using the ordinary least squares method, and the C is a constant.

Assuming $e^{\int a dt} = e^{at+w}$, where w is a constant, then Eq. (7) can be equivalent to:

$$x^{(1)}(t)\, e^{at+w} - C = \int (bt^{\alpha} + c)\, e^{at+w} dt \tag{8}$$

Letting $F(t) = x^{(1)}(t)\, e^{at+w} - C$ and $G(t) = \int (bt^{\alpha} + c)\, e^{at+w} dt$, then $F(t) = G(t)$ for any t. according to the classical grey system theory, appropriate selections of the initial condition is essential to access to particular solution formula of Eq. (2). Theoretically, this initial condition value could be randomly assigned to any specific value. However, for convenience and simplification purposes, the first data point in the collected observations is usually considered as the first choice, i.e., $x^{(1)}(t)\big|_{t=1} = x^{(0)}(1)$. Subsequently, Setting $t = 1$ and $t = t_0 (t_0 > 1)$, then one can obtain:

$$F(t_0) - F(1) = G(t_0) - G(1) \tag{9}$$

As a consequence, the lower bound of the definite integral is determined by choosing the initial condition values. Then, substituting $F(1) = x^{(1)}(1)e^{at_0+w} - C$ into Eq. (9), this equation can be equally transferred into:

$$x^{(1)}(t_0)\, e^{at_0+w} - x^{(1)}(1)\, e^{a+w} = \int_{1}^{t_0} (bt^{\alpha} + c)\, e^{at+w} dt \tag{10}$$

Integrating and simplifying Eq. (10), one will have

$$x^{(1)}(t_0)\, e^{at_0} - x^{(1)}(1)\, e^{a} = \int_{1}^{t_0} (bt^{\alpha} + c)\, e^{at} dt \tag{11}$$

Integrating and solving Eq. (11), one can get the simplified time response function for the $GM(1,1,t^{\alpha})$ whitening model (abbreviated as $OGM(1,1,t^{\alpha})$):

$$x^{(1)}(t_0) = e^{-at_0}\int_{1}^{t_0} bt^{\alpha} e^{at} dt + \frac{c}{a} - \frac{c}{a}e^{-a(t_0-1)} + x^{(1)}(1)\, e^{-a(t_0-1)} \tag{12}$$

Replacing the t_0 with the variable t, the generalized version of Eq. (12) is:

$$x^{(1)}(t) = e^{-at}\int_{1}^{t} b\tau^{\alpha} e^{a\tau} d\tau + \frac{c}{a} - \frac{c}{a}e^{-a(t-1)} + x^{(1)}(1)\, e^{-a(t-1)} \tag{13}$$

Discretizing Eq. (13), then one can obtain the corresponding discrete version of the time response function, seen in Eq. (5). Subsequently, by using the $1-IAGO$ technique, the forecasting function in the original domain can be given, seen in Eq. (6). Thus, Theorem 3 has been proved. Subsequently, by utilizing Eqs. (5) and (6), one will be able to generate forecasts in any practical studies under the premise of a known parameter α.

Observing Eqs. (4) and (5), one can find that the simplified time response function for the $OGM(1,1,t^{\alpha})$ model is not only related to the initial value but also associated with the a, c, and time k. It is worth mentioning that when the parameter α is an integer, the primitive function of $\int_1^t b\tau^{\alpha} e^{a\tau} d\tau$ exists and projections can be generated by using Eq. (5). However, when the parameter α is a non-integer, the primitive function of $\int_1^t b\tau^{\alpha} e^{a\tau} d\tau$ does not exist, which means that one cannot evaluate this definite integral. To this end, one can estimate this definite integral based on the dispersion sum technique, namely $\int_1^t b\tau^{\alpha} e^{a\tau} d\tau \approx \sum_1^{\tau=t} b\tau^{\alpha} e^{a\tau} \Delta\tau$, where $\Delta\tau = 0.0001$. In general, the proposed $OGM(1,1,t^{\alpha})$ model can produce any forecasts no matter what kinds of the parameter α is, which expands the application fields of the conventional grey model in reference Qian et al. (2012).

3.2. The solutions to the parameter α in the proposed model

As revealed in Section 3.1, parameters (a,b,c) are essential for the time response function. Thus, a,b,c are called principle parameters, and (a,b,c) is noted as the I-order parameter packet of the $OGM(1,1,t^{\alpha})$ model, abbreviated $\mathbf{P_I} = (a,b,c)^T$. By using the ordinary least square method, explained in Section 2, one can obtain $\mathbf{P_I} = (a,b,c)^T = \left(\mathbf{B^T B}\right)^{-1}\mathbf{B^T Y}$, where

$$\mathbf{B} = \begin{pmatrix} -z^{(1)}(2) & 2^{\alpha} & 1 \\ -z^{(1)}(3) & 3^{\alpha} & 1 \\ \vdots & \vdots & \vdots \\ -z^{(1)}(n) & n^{\alpha} & 1 \end{pmatrix} \mathbf{Y} = \begin{pmatrix} x^{(0)}(2) \\ x^{(0)}(3) \\ \vdots \\ x^{(0)}(n) \end{pmatrix} \tag{14}$$

For estimating $\mathbf{P_I} = (a, b, c)^T$, some component parameters are introduced, which are noted as the II-order parameter packet,abbreviated as $\mathbf{P_{II}} = (A, B, C, D, E, F, G, H)^T$, where $A = \sum_{k=2}^{n} z^{(1)}(k)$, $B = \sum_{k=2}^{n} k^{\alpha}$, $C = \sum_{k=2}^{n} (z^{(1)}(k))^2$, $D = \sum_{k=2}^{n} k^{2\alpha}$, $E = \sum_{k=2}^{n} z^{(1)}(k) \times k^{\alpha}$, $F = \sum_{k=2}^{n} -z^{(1)}(k) \times X^{(0)}(k)$, $G = \sum_{k=2}^{n} k^{\alpha} \times X^{(0)}(k)$, and $H = \sum_{k=2}^{n} X^{(0)}(k)$. Besides, the actual values of the basic variables compose the III-order parameter packet, abbreviated as $\mathbf{P_{III}} = \left(x^{(0)}(k), z^{(1)}(k)\right)^T$.

Based on the three orders parameter packet, one can obtain

$$\left(\mathbf{B^T B}\right)^{-1} = \frac{\begin{pmatrix} (n-1)D - B^2 & (n-1)E - AB & AD - BE \\ (n-1)E - AB & (n-1)C - A^2 & AE - BC \\ AD - BE & AE - BC & CD - E^2 \end{pmatrix}}{(n-1)CD + 2ABE - A^2 D - (n-1)E^2 - B^2 C}, \tag{15}$$

$$\mathbf{B^T Y} = \begin{pmatrix} F & G & H \end{pmatrix}^T. \tag{16}$$

Subsequently, the parameter packages of the $OGM(1,1,t^{\alpha})$ model can be expressed as

$$\mathbf{P_I} = (a, b, c)^T = \frac{\begin{pmatrix} ((n-1)D - B^2)F + ((n-1)C - A^2)G + (AD - BE)H \\ ((n-1)E - AB)F + ((n-1)E - AB)G + (AE - BC)H \\ (AD - BE)F + (AE - BC)G + (CD - E^2)H \end{pmatrix}}{(n-1)CD + 2ABE - A^2 D - (n-1)E^2 - B^2 C} \tag{17}$$

By using the parameter packages, the detailed estimations of the principle parameters a, b, c can be easily understood by a reader who has little expert knowledge. In addition to the principle parameters, the parameter α also plays an essential role in obtaining an accurate time response function (explained in Section 3.1). Due to the complexity of estimating parameter α, its optimal value is calculated by minimizing the mean relative squared error (MRSE) between the forecasted and actual data points. For this, the objective function can be expressed as:

$$\begin{aligned} &Min \quad MAPE = \frac{1}{n}\sum_{k=1}^{n}\left|1 - \frac{\hat{x}^{(0)}(k)}{x^{(0)}(k)}\right| \\ &s.t. \begin{cases} (a, b, c)^T = \left(\mathbf{B^T B}\right)^{-1}\mathbf{B^T Y} \\ x^{(1)}(k) = e^{-at}\int_1^k b\tau^{\alpha} e^{a\tau} d\tau + \frac{c}{a} - \frac{c}{a}e^{-a(k-1)} + x^{(1)}(1)\, e^{-a(k-1)} \\ \hat{x}^{(0)}(k+1) = \hat{x}^{(1)}(k+1) - \hat{x}^{(1)}(k) \\ k = 1, 2, \dots, n \end{cases} \end{aligned} \tag{18}$$

where $\hat{x}^{(0)}(k)$ represents the forecasted values, $x^{(0)}(k)$ stands for the actual observations, and n is the total number of the input data points.

To obtain the optimum parameter α in the $OGM(1,1,t^{\alpha})$ model, the objective function in Eq. (18) should approach its minimum value to an extreme. Owing to its nonlinear features, the objective function can be solved by using an intelligent algorithm, such as *PSO* for solving some complicated optimization problems. The main procedures of *PSO* are outlined as follows

Step 1: Define the time-power parameter α and establish a fitness function for each particle. According to the objective function mentioned above, the expression of the fitness function can be defined as

$$Fitness[P(i,j)]^T = \frac{1}{n}\sum_{k=1}^{n}\left|1 - \frac{\hat{x}^{(0)}(k)}{x^{(0)}(k)}\right| \tag{19}$$

Step 2: Set parameters. Before implementing the *PSO*, several parameters need to be defined. $P(i,j)$ represents the current position of the jth particle in the ith iteration. $Pbest(i)$ records the position when getting minimal fitness in the ith iteration. Moreover, $Gbest$ records the global elite position around the whole search space and updates new positions when $Gbest > Pbest(i)$.

Step 3: Initialize the position and speed randomly for each particle and renew positions according to the rules. The particle velocity formula in the first iteration is given as follows:

$$v_{k+1} = d_1(pbest_k - y_k)Q_1 + d_2(gbest_k - y_k)Q_2 \tag{20}$$

where Q_1 and Q_2 are random variables belonging to $[0,1]$. Additionally, d_1 and d_2 are the acceleration factors. Then, the rules for updating particles' position satisfy $P(i, j+1) = P(i,j) + v_k$. And the particle velocities for the next iteration can be reckoned as

$$v_{k+1} = hv_k + d_1(pbest_k - y_k)Q_1 + d_2(gbest_k - y_k)Q_2 \tag{21}$$

where h represents an inertia weight for adjusting convergence rate.

Step 4: Obtain the optimal solutions when two termination criteria are reached: (1) approaching the minimal fitness value mathematically. (2) reaching the maximum number of iterations.

In general, the programming problem in the $OGM(1,1,t^{\alpha})$ model can be solved by using the *PSO* technique to obtain the optimal parameter α based on the characteristics of the system sequence. Subsequently, once one obtains all the parameters a, b, c in Section 2 and the time-power parameter α in Section 3.2, the $OGM(1,1,t^{\alpha})$ model will generate projections.

Besides, for demonstrating the reliability of the proposed model, probability density prediction based on Monte Carlo Simulation is carried out in this paper. As illustrated in the above procedures for the *PSO*, one need to initialize the position and speed for each particle by randomly generating a certain value for the time parameter α, which means that the optimal value of α for the end-up phrase may be different when setting diverse initialized values at the beginning phrase. In other words, the optimal value of α is possibly different for each iterative operation of the *PSO*. As a consequence, the estimated values of parameters a, b, c and their corresponding predicted values of roadbed settlements diverse significantly. Thus, the reliability and accuracy of the proposed model should be ensured for future estimations of subgrade settlements. To this end, the Monte Carlo Simulation Test is necessary for ensuring its forecasting repeatability. In this simulation study, tests on predicting the settlement of soft-clay subgrade on the expressway will be taken for 1000 times. Correspondingly, one will obtain 1000 parameter estimates and predicted sequences, respectively. In conclusion, the proposed model optimizes the prediction model, which will lay the foundation for the follow-up prediction work.

3.3. Probability density prediction based on the proposed model

As described in the previous subsection, due to the diverse parameter values of α for each iterative operation of the *PSO*, the following estimated values of a, b, and c (explained in Theorem 1) and the forecasted values generated by the corresponding simplified time response function in Eq. (10) vary for each time. Therefore, the reliability and accuracy of the proposed model should be ensured for future estimations. To deal with such a situation, the method of probability density prediction is introduced in this paper, which can provide more information about the parameters and forecasting data of the proposed model. In this method, the kernel density estimation (*KDE*) technique is employed to calculate the probability density.

As researchers revealed, *KDE* is a nonparametric method for estimating the probability density functions (Sheather and Jones, 1991) and can fit observations to simulate a true probability distribution by a smooth spike function (He et al., 2019). One takes the different values of the time-power parameter α as an example for explanations. For the Monte Carlo Simulation, 1000 values of the time-power parameter α can be obtained by using the PSO, which can be represented as $P = (p_1, p_2, \dots, p_{1000})$. The estimated value $p_i (i = 1, 2, \dots, 1000)$ is used as the input values of the kernel function. Additionally, based on the selected bandwidth for the probability density analysis, the probability density functions of the time-power parameter α are obtained eventually. Assuming that $p_1, p_2, \dots, p_{1000}$ are the independent random samples with the identical distribution, the probability density function at any one point is denoted as f, whose formula is expressed below.

$$f_h(\tau) = \frac{1}{n}\sum_{i=1}^{n} K_h(\tau - p_i) = \frac{1}{nh}\sum_{i=1}^{n} K_h\left(\frac{\tau - p_i}{h}\right) \tag{22}$$

Table 1
The criterion for *MAPE* values.

MAPE (%)	Predictive performance	*MAPE* (%)	Predictive performance
<10	High precision	10–20	Good precision
20–50	Reasonable precision	>50	Weak precision

In this equation, h represents the bandwidth that is a smoothing parameter requiring a self-actuated setting, n stands for the sample size (here $n = 1000$), and $K(\cdot)$ is a non-negative kernel function. In this paper, the Gaussian kernel function is chosen as the representative of the kernel function, whose expression is displayed as follows.

$$K(u) = \frac{1}{\sqrt{2\pi}} e^{-0.5u^2} \tag{23}$$

In addition to the time-power parameter α, the probability density of the parameters a, b, c and their corresponding predicted values of roadbed settlements can also be measured by using a similar way. The flow chart of the algorithm in this paper is presented in Fig. 2.

3.4. Measurement of accuracy

In this paper, to verify the reliability and applicability of the newly-designed technique, three critical evaluation indicators are put forward:

The Absolute Percentage Error (APE) and Mean Absolute Percentage Error ($MAPE$) are utilized for analysis to point forecasted results. The criterion of accuracy grades for $MAPE$ values is exhibited in Table 1. They are defined as follows.

$$\mathrm{APE}(k) = \left| \left(\hat{x}^{(0)}(k) - x^{(0)}(k) \right) / x^{(0)}(k) \right| \times 100\% \tag{24}$$

$$MAPE = \frac{1}{n-1} \sum_{k=2}^{n} APE(k) \tag{25}$$

The Root Mean Squared Error ($RMSE$) is used to reflect the deviation degree of the forecasted values compared with the original ones. This statistical indicator is calculated by using the following Eq. (22).

$$RMSE = \sqrt{\frac{1}{n-1} \sum_{k=2}^{n} \left(\hat{x}^{(0)}(k) - x^{(0)}(k) \right)^2} \tag{26}$$

In these above three indicators, n is the total number of data points for the settlements of soft clay to be predicted, $x^{(0)}(k)$ and $\hat{x}^{(0)}(k)$ stand for the original and forecasted values at the time k, respectively.

Additionally, the Percentage Error Analysis (*PEA*) is introduced to compare the forecasting performance between two competitors, whose formula is presented in Eq. (27)

$$PEA = \left| \frac{MAPE_1 - MAPE_2}{MAPE_2} \right| \times 100\% \tag{27}$$

where $MAPE_1$ and $MAPE_2$ represent the MAPE values produced by models 1 and 2, respectively. By using this formula, one can measure the forecasting performance of the model 1 compared with the model 2, thereby effectively identifying the best forecasting technique among all the competitors.

4. Empirical analysis

In this section, two experiments are conducted to verify the efficacy and reliability of the newly proposed models. To be specific, the first one is designed to show that the simplified time response function in the proposed model is consistent with that of the conventional model, verifying its universality and applicability. The other one is to demonstrate the expanded application fields of the new model, which is predicting the settlements of soft-clay subgrade on the expressway.

4.1. Verification of the simplified time response function for the proposed model

As introduced by Qian et al. (2012), he provided the accurate time response function of the conventional $GM(1,1,t^\alpha)$ model when $\alpha = 1$ and $\alpha = 2$, which can be outlined as follows:

When $\alpha = 1$, the time response function of the $GM(1,1,t^1)$ model is:

$$x^{(1)}(t) = \left(x^{(1)}(1) - \frac{b}{a} - \frac{ac-b}{a^2} \right) e^{-a(t-1)} + \frac{b}{a}t + \frac{ac-b}{a^2} \tag{28}$$

When $\alpha = 2$, the time response function of the $GM(1,1,t^2)$ model is:

$$x^{(1)}(t) = \left(x^{(1)}(1) - \frac{a^2b + a^2c - 2ab + 2b}{a^2} \right) e^{-a(t-1)} + \frac{b}{a}t^2 - \frac{b}{a^2}t + \frac{a^2c + 2b}{a^2} \tag{29}$$

For the purpose of verifying the efficacy and adaptability of the proposed model, one would randomly generate a sequence, based on which two versions of the $GM(1,1,t^\alpha)$ models are established for generating forecasts with the time parameter $\alpha = 1$ and $\alpha = 2$, respectively. Subsequently, two groups of forecasted sequences will be obtained: one is provided by the original time response function from the conventional $GM(1,1,t^\alpha)$ model, and the other one is generated by the simplified time response function from the $OGM(1,1,t^\alpha)$ model. Comparing these two groups of generated sequences, one can conclude: if little deviations exist between these two series, the $OGM(1,1,t^\alpha)$ having a simplified time response function is effective and practicable for replacing the traditional one. Otherwise, the proposed model fails to unify the conventional grey model.

One randomly generates a non-homogenous sequence by using the equation: $X(k) = e^{0.1k} + 2, k = 1, 2, \ldots, 6$. Based on these data points, the $OGM(1,1,t^\alpha)$ and conventional $GM(1,1,t^\alpha)$ models are established to check whether they are equivalent for generating forecasts, whose numerical results are listed in Table 2.

From Table 2, it is seen that two groups of the forecasted sequences are provided. To be specific, one is generated by the $GM(1,1,t^1)$ and $OGM(1,1,t^1)$ models, and the other one is predicted by the $GM(1,1,t^2)$ and $OGM(1,1,t^2)$ models. Intuitively observed from Table 2, one find that the values from the $OGM(1,1,t^\alpha)$ model are almost identical with those from the conventional $GM(1,1,t^\alpha)$ model no matter $\alpha = 1$ or $\alpha = 2$. These findings illustrate that the simplified time response function from the proposed model (one of the greatest contributions in this paper) can effectively produce the same projections as the original one from the classic model. Additionally, it is worth mentioning that the proposed model can model diverse sequences if the time-power parameter α equals to different values. Therefore, the simplified time response function for the new model is correct and effective, which has much broader application areas compared with the conventional $GM(1,1,t^\alpha)$ model.

4.2. Settlement prediction of soft-clay subgrade for an expressway

To demonstrate the reliability and efficacy of the proposed model, another case study on predicting settlement of the soft-clay foundation on the highway is conducted by comparing it with several forecasting techniques. Most importantly, besides the three measurement indicators, explained in Section 3.3, an alternative analysis method of the probability density prediction is initially introduced in this case, which provides a reliable foundation for supporting the superior performance of the proposed model. Then, the actual case about predicting settlement of soft-clay subgrade on the highway comes below.

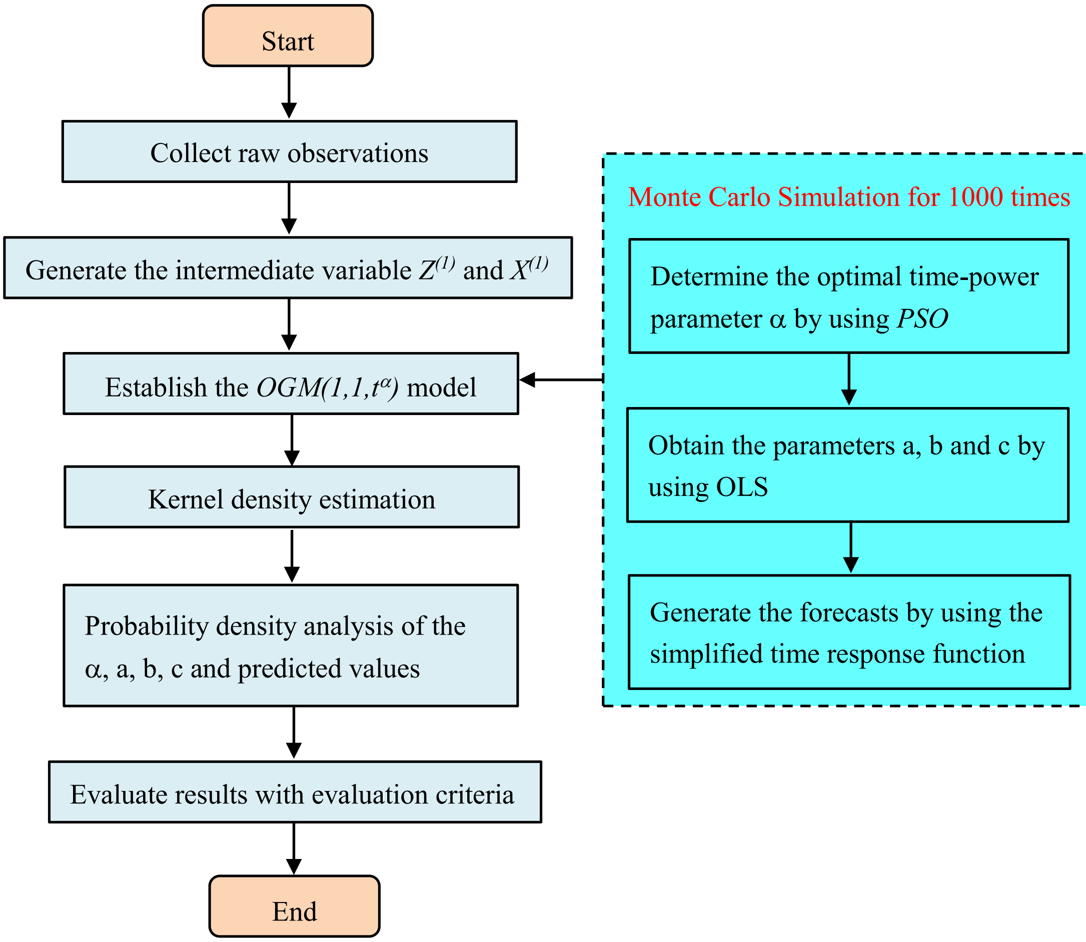

Fig. 2. The flow chart of the proposed model and its probability density prediction.

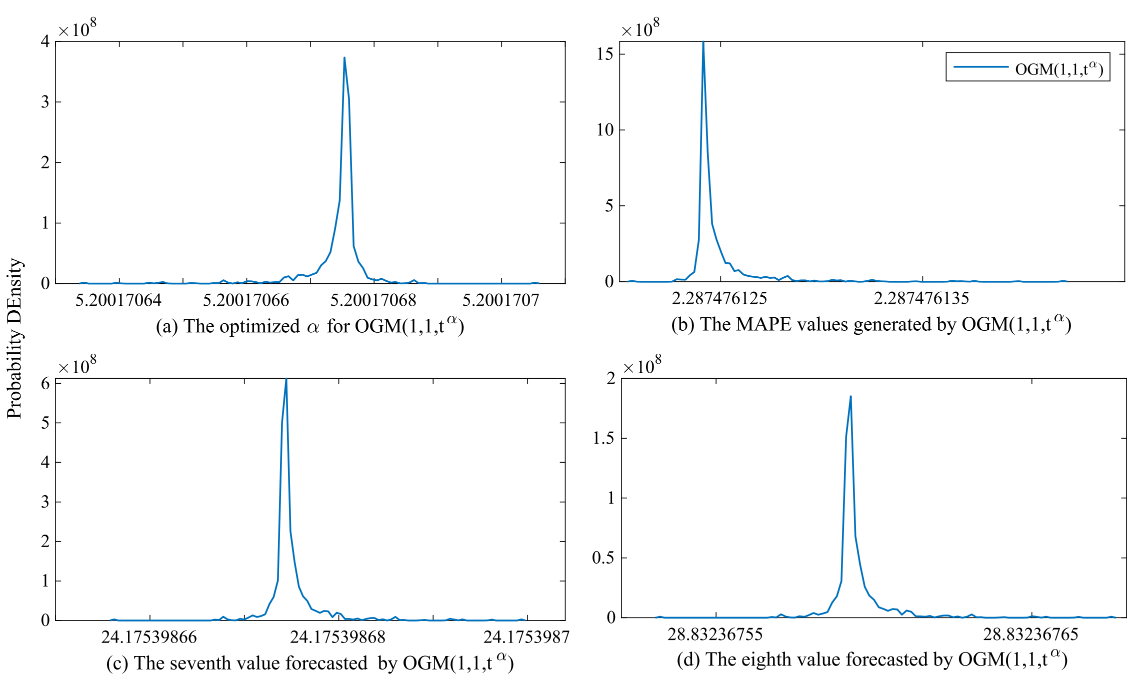

Fig. 3. The probability density curves concerning the generated results of $OGM(1,1,t^{\alpha})$.

Table 2
Numerical results provided by the *OGM (1,1,t^α)* and conventional *GM (1,1,t^α)* models.

X	Original sequence						Parameters		
	3.1052	3.2214	3.3499	3.4918	3.6487	3.8221	a	b	c
GM(1,1,t^1)	3.1052	3.3254	3.4647	3.6186	3.7887	3.9766	−0.0999	−0.1998	3.1499
OGM(1,1,t^1)	3.1052	3.3251	3.4646	3.6185	3.7886	3.9765	−0.0999	−0.1998	3.1499
Error	0	0.0003	0.0001	0.0001	0.0001	0.0001	0	0	0
GM(1,1,t^2)	3.1052	3.2126	3.3349	3.4715	3.6229	3.7895	−0.0311	0.0052	3.0545
OGM(1,1,t^2)	3.1052	3.2126	3.3349	3.4715	3.6229	3.7896	−0.0311	0.0052	3.0545
Error	0	0	0	0	0	−0.0001	0	0	0

Table 3
The designations and time response functions for the five nonlinear grey models.

Models	Model designations	The time response functions
M1: $GM(1,1,t)$	$\frac{dx^{(1)}}{dt}+ax^{(1)}=bt+c$	$\hat{x}^{(1)}(k)=48.36\cdot e^{0.19(k-1)}-5.34k-39.72$
M2: $GM(1,1,t^2)$	$\frac{dx^{(1)}}{dt}+ax^{(1)}=bt^2+c$	$\hat{x}^{(1)}(k)=-33.46\cdot e^{0.19(k-1)}+5.34k^2-13.29k+47.61$
M3: $GPM(1,1)$	$\frac{dx^{(1)}}{dt}+ax^{(1)}=b(x^{(1)})^\alpha$	$\hat{x}^{(1)}(k)=[14.70\cdot e^{0.17(k-1)}-12.24]^{1.33}$
M4: $NDGM(1,1)$	$\hat{x}^{(1)}(k+1)=\beta_1\hat{x}^{(1)}(k)+\beta_2 k+\beta_3$	$\hat{x}^{(1)}(k+1)=1.21\cdot\hat{x}^{(1)}(k)+1.20k+3.64$
M5 and *M6*: $OGM(1,1,t^\alpha)$	$\frac{dx^{(1)}}{dt}+ax^{(1)}=bt^\alpha+c$	$\hat{x}^{(1)}(k)=13.50\cdot e^{0.34(k-1)}-10.2+e^{0.34k}\sum_{1}^{\tau=k}-0.0002\cdot\tau^{5.20}e^{-0.34\tau}\Delta\tau,\ \Delta\tau=0.0001$

Note: the median and mean values for the parameter α in $GM(1,1,t^\alpha)$ obtained by Monte Carlo Simulation are the same, thus the generated time response functions for *M5* and *M6* are the same.

4.2.1. Data collection

In recent years, the settlement forecasting of soft-clay highway roadbed is one of the prominent problems in engineering. As the soft clay has the features of high compressibility, high water content, low strength, and poor permeability, it will result in unpredictably abnormal and large settlements, which weaken the highways' structural and serviceability performance (Müthing et al., 2018). Therefore, an accurate prediction technique is essential as well as a countermeasure for settlements to ensure each expressway fulfills its sufficient functions.

Besides, as the settlement of soft clay usually changes with time and is influenced by many uncertain factors, such as traffic loading, the trends of the collected data normally appear to be nonlinear with a dynamic growth rate. Thus, accurate prediction methods are necessary to deal with such situations, which can resist the effects of the above factors. Recently, more and more emerging techniques have been employed to forecast the settlements, such as the grey prediction model (Qian et al., 2012) and the Finite Element Model (Müthing et al., 2018). To accurately predict the settlement, the $OGM(1,1,t^\alpha)$ model is utilized in comparison with several competing models.

In this study, the original observations are collected from Qian et al. (2012), which have eight data points in Table 3. For the purpose of testing the fitted and predicted ability of the competing models, six observations are used as input data for model calibration as well as checking their fitted capability. Then, the remaining two data points are employed for testing their forecasting performance.

4.2.2. Model calibration for the proposed model

In this section, in order to verify the accuracy and reliability of the $OGM(1,1,t^\alpha)$ model, probability density prediction based on the Monte Carlo Simulation study is carried out in this paper, whose procedures are presented in Fig. 4. Following these steps, the probability density curves of the optimized parameter α, the corresponding *MAPE* values, as well as the forecasted results of the seventh and eighth data in Monte Carlo Simulation are depicted in Fig. 3.

In Fig. 3(a), the value of the abscissa represents the optimized values of the time-power parameter α solved by PSO in the $OGM(1,1,t^\alpha)$ model, and the ordinate stands for the probability density that can reflect the distribution of possible values of the parameter α. Similarly, the probability density curve of the generated corresponding *MAPE* values is depicted in Fig. 3(b). From these two figures, the following conclusions can be drawn. The two probability density curves all appear to have a single peak during the whole iteration process. For the time-power parameter α, its estimated values appear at the intervals of $[5,6]$, and its mean and median values are around 5.20 with extremely high probability. This phenomenon indicates that the proposed model incorporated with PSO has strong stability and high reliability because the estimated values of the time-power parameter are almost in the vicinity of the highest probability point in the probability density curve. As shown in Fig. 3(b), based on the solutions of the parameter α, the generated *MAPE* values demonstrate analogous distribution features with slight fluctuations, and most values are distributed around 2.287 at the single peak. Therefore, it is easy to find that the $OGM(1,1,t^\alpha)$ model has consecutive and slippy probability density curves of the modeling parameters and corresponding errors, indicating that its generated forecasts have a significant likelihood of getting closer to the actual observations.

Subsequent to the estimations of all these above parameters, the probability density curves of predicted values by using the $OGM(1,1,t^\alpha)$ model in Monte Carlo Simulation are presented in Fig. 3(c) and (d). Similar to the findings of the parameters, the probability density of the seventh and eighth forecasted values also show a curvilinear change with one peak. To be specific, the forecasted values of the seventh datapoint are almost in the vicinity of the highest probability point in the probability density curve, approximating 24.18, which is highly close to the actual value of 23.80. For the eighth data point, its predicted values of 28.83 are getting closer to the actual observation of 28.60 with the highest probability. For the purpose of further revealing its effectiveness, the mean and median values of the projections in the fitted and predicted domain are provided in Table 3. In general, the probability density curve can provide more detailed information for settlement predictions of soft-clay subgrade and present new ideas and methods for demonstrating the reliability and stability of the proposed model.

4.2.3. Comparative analysis

In order to further illustrate the efficacy of the newly designed model, a range of grey and non-grey models are included in this paper.

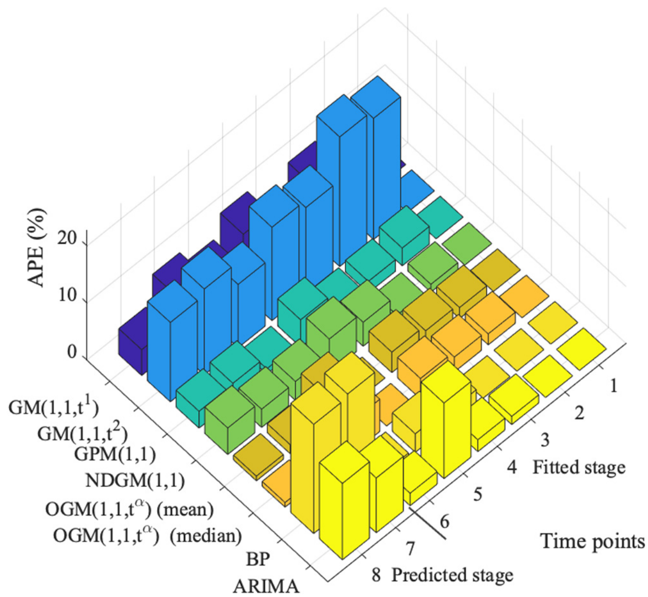

Fig. 4. APEs for the eight competing models in both fitted and predicted stages.

As the representations of the probability density prediction for the $OGM(1,1,t^{\alpha})$ model, the mean and median values of the predicted values for 1000 times are chosen for explanation purposes, which are remarked as M_5 and M_6, respectively. For the other competing models, the $GM(1,1,t^1)$, $GM(1,1,t^2)$, $GPM(1,1)$, $NDGM(1,1)$, $BPNN$, and $ARIMA(1,1,0)$ models are noted as M_1, M_2, M_3, M_4, M_7, and M_8, respectively. The model designations and solved time response functions for these competing nonlinear grey models are presented in Table 3. Besides, the estimated values and corresponding errors in the fitted and predicted stages are provided in Table 4. For visual presentation, Fig. 4 is depicted to show the *APE* values at the observed time points for the eight competing models.

According to the experimental results in Table 4 and Fig. 4, the mean and median values of the forecasts generated by using the $OGM(1,1,t^{\alpha})$ model are much closer to the valid values in both fitted and predicted domains, compared with those values obtained by the $GM(1,1,t^1)$, $GM(1,1,t^2)$, $GPM(1,1)$, and $NDGM(1,1)$ models. Specifically, compared with these four grey models, the proposed model can increase the forecasting precision with the percentage of 68.84%, 83.95%, 10.10%, and 11.92% in the fitting stage, respectively, according to the PEA results. Besides, in the predicted period, the novel model can produce forecasts with the improved accuracy percentages of 82.75%, 91.76%, 60.73%, and 69.49%, respectively. Thereby, the $OGM(1,1,t^{\alpha})$ model turns to be more effective and practical than the grey benchmarks.

Additionally, it can be observed from Fig. 4 that $OGM(1,1,t^{\alpha})$ exhibits the stable and extraordinary simulation capability with the *APE* values at all the time points distributed at a low level. Moreover, in the predicted period, the proposed $OGM(1,1,t^{\alpha})$ model exhibits the superlative forecasting precision with the smallest *MAPE* value of 1.19% and *RMSE* of 0.31 by contrast with the other benchmarks. Besides, it can be indicated from Table 3 that the proposed model is provided with a concise structure and has more robust adaptability for dealing with non-linear issues by virtue of the adaptive time-power parameter α.

From the above analysis and the generated results presented in Tables 3 and 4, the superiority of $GM(1,1,t^{\alpha})$ can be interpreted from the following two aspects. On the one hand, the traditional $GM(1,1,t^{\alpha})$ model has comparatively narrow application fields owing to its restricted time-power parameter α(usually $\alpha = 0, 1 or 2$). To enhance the model adaptability, the generalized solution to the proposed model with arbitrary values including both integers and non-integers assigned to the time-power coefficient α is provided in Theorem 3. Therefore, the intelligent coefficient α enables $GM(1,1,t^{\alpha})$ to extract the nonlinear features in diverse time series, thereby broadening the application fields of the convention model. On the other hand, the proposed model is refined with a simple structure that may accelerate the forecasting speed and seem to be more explainable in contrast with $GPM(1,1)$ and $NDGM(1,1)$ that have relatively more complicated structures. Furthermore, by incorporating PSO to solve the time-power coefficient, the forecasted results of the proposed model can achieve high precision with strong robustness.

Furthermore, compared to the non-grey models, namely $BPNN$ and $ARIMA(1,1,0)$, the proposed model still achieves better performance in the predicted phase, although these two non-grey models have slightly lower *MAPE* and *RMSE* values in the fitted area. To be specific, the $BPNN$ and $ARIMA(1,1,0)$ models lost several predicted values in their first three and two data points, which cause the lower *MAPE* and *RMSE* values. However, they both produce larger *MAPE* and *RMSE* values in the predicted domain, which means their projections deviate far from the true values. Moreover, from the PEA perspective, the forecasting accuracy of this new model can be increased by 93.86% and 89.63%, respectively, in the predicted period when comparing with the $BPNN$, and $ARIMA(1,1,0)$ models. For these two non-grey models, they are highly dependent on the amount of training data. Otherwise, they may produce unacceptable errors in real applications. Therefore, compared with the $BPNN$ and $ARIMA(1,1,0)$, the proposed model shows much more stability and reliability while dealing with insufficient data. It can be chosen as an alternative forecasting technique for predicting settlements of soft-clay subgrade on an expressway.

In general, the $OGM(1,1,t^{\alpha})$ model achieves better performance than the other grey and non-grey competitors in terms of the measuring indicators *MAPE* and *RMSE*. Moreover, this proposed model also obtain strong stability and high reliability in both fitted and predicted domains (explained by using probability density analysis in Section 4.2.2), which justifies the superiority of the newly designed model.

5. Conclusions

Aiming at solving the nonlinear forecasting issues in real applications, this paper proposed an optimized time-power based grey prediction model -$OGM(1,1,t^{\alpha})$. Initially, the advantages and disadvantages of the conventional $GM(1,1,t^{\alpha})$ model are analyzed comprehensively. Secondly, to eliminate the above drawbacks, a simplified time response function is put forward, enabling the new model to adapt to various issues with diverse values of the time-power parameter. Thirdly, for accurately determining the optimal time-power parameter, the PSO is employed, and the parameter packages are provided for illustrating the detailed process of parameter estimations. Lastly, two experiments are conducted to demonstrate the efficacy and reliability of the proposed model. Our main findings are:

(1) The simplified solution to the differential equation is effective and practical to overcome the limitations of the previous model. This improvement enables the proposed model to solve various nonlinear issues with strong adaptability because the time-power parameter can take any values based on the characteristics of the modeling sequences. Moreover, this new model can significantly improve the forecasting accuracy with the support of PSO.

(2) With respect to the promising method of probability density analysis, the true values of the parameters and predicted values are almost all in the vicinity of the highest probability point, which shows strong stability and high reliability of the proposed model. Thus, the probability density curve can not only provide more detailed information for settlement predictions of soft-clay subgrade but also present new ideas and methods for demonstrating the performance of the proposed model.

Table 4
Fitted and forecasted values of four competing models for settlements prediction of soft-clay subgrade on an expressway [Unit: cm].

$x^{(0)}(k)$	M_1		M_2		M_3		M_4		M_5		M_6		M_7		M_8	
	$\hat{x}^{(0)}(k)$	APE	$\hat{x}^{(0)}(k)$	APE	$\hat{x}^{(0)}(k)$	APE	$\hat{x}^{(0)}(k)$	APE	$\hat{x}^{(0)}(k)$	APE	$\hat{x}^{(0)}(k)$	APE	$\hat{x}^{(0)}(k)$	APE	$\hat{x}^{(0)}(k)$	APE
The Fitted stage																
3.3	3.30	0.00	3.30	0.00	3.30	0.00	3.30	0.00	3.30	0.00	3.30	0.00	–	–	3.30	–
5.6	4.94	11.87	4.40	21.19	5.43	3.11	5.52	1.38	5.49	1.92	5.49	1.92	–	–	5.60	–
7.9	7.12	9.89	6.11	22.67	7.74	2.02	7.96	0.72	7.70	2.59	7.70	2.59	–	–	7.79	1.43
10.3	9.77	5.18	8.82	15.05	10.47	1.68	10.73	4.20	10.67	3.58	10.67	3.58	10.55	2.43	10.09	2.07
14.5	12.98	10.51	12.20	16.33	13.81	4.76	13.63	6.00	14.50	0.00	14.50	0.00	14.97	3.24	12.58	13.23
18.1	16.87	6.80	16.05	11.07	17.93	0.92	18.70	3.31	19.12	5.64	19.12	5.64	18.25	0.83	17.71	2.18
MAPE (%)		7.35		14.27		2.08		2.60		2.29		2.29		2.17		4.72
RMSE		0.93		1.65		0.31		0.47		0.45		0.45		0.32		0.99
The predicted stage																
23.8	21.59	9.29	20.23	15.00	23.06	3.11	23.04	3.18	24.18	1.58	24.18	1.58	19.12	19.66	21.52	9.58
28.6	27.31	4.51	24.63	13.88	29.44	2.95	29.92	4.63	28.83	0.81	28.83	0.81	23.14	19.09	24.78	13.36
MAPE (%)		6.90		14.44		3.03		3.90		1.19		1.19		19.38		11.47
RMSE		1.81		3.78		0.74		1.08		0.31		0.31		5.09		3.15

(3) In addition to the probability density curve for measuring the stability and reliability of the proposed model, four measuring statistical indicators, namely *APE, MAPE*, *RMSE*, and *PER,* are chosen for comparative analysis. Results show that the $OGM(1,1,t^{\alpha})$ model can provide more accurate projections and reduce the uncertainty of settlement prediction of soft-clay subgrade on an expressway. Consequently, the proposed model can be selected as the optimal technique for future estimations of the settlements.

In general, the idea to design the novel model has provided a new angle to optimize the conventional $GM(1,1,t^{\alpha})$ model. Much work will be carried out in the future. On the one hand, more improvements on this new model can be conducted to enhance further its forecasting capabilities, such as optimizing the background value, the initial condition, and the model structure. On the other hand, more non-linear forecasting issues in other domains, such as industrial and economic fields, can be solved by using this proposed model. Thus, the proposed model has potentially enriched the grey system theory.

CRediT authorship contribution statement

Keyong Wan: Writing – original draft, Conceptualization, Investigation, Methodology. **Bin Li:** Supervision, Reviewing and editing. **Weijie Zhou:** Programming on software, Data handling, Investigation, Reviewing and editing. **Haicheng Zhu:** Writing - revised manuscript, Reviewing and editing. **Song Ding:** Investigation, Validation, Methodology, Writing - revised manuscript, Reviewing and editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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