Computational Economics (2019) 53:457–478
https://doi.org/10.1007/s10614-018-9834-5

The Complexion of Multi-period Stackelberg Triopoly Game with Bounded Rationality

Yu Yu[1] · **Weisheng Yu**[2]

Accepted: 28 June 2018 / Published online: 6 July 2018

Abstract

Stackelberg model is a dynamic model, in which two players with different scales and power players act sequentially. However, there are few literatures that apply complex oligopoly dynamics theory in this model. In this paper, based on a traditional Stackelberg model, we improve the model in Peng and Lu (Appl Math Comput 271:259–268, 2015) and construct a multi-period Stackelberg triopoly game model. One leader firm and two followers with bounded rationality behavior are considered. The leader's decision-making variable, which is simplified as a constant in Peng and Lu's paper, is observed by the followers in stage 1 in every period in this model. We arrive at the conclusion that the leader would have the first-move advantage even when the players adopt a gradient output adjustment process in a multi-period Stackelberg triopoly game model. The speeds of output adjustment form a three-dimensional stability region. In the equilibrium state, the outputs of the followers are one-third of the leader's. With adjustment speed of the leader increasing, Stackelberg equilibrium would be broken at a certain point. The effect of adjustment speed on speed of convergence to equilibrium is also analyzed. Theoretical result and numerical simulation both demonstrate that the speed converging to equilibrium is slowing when the Lyapunov exponent increases. Strange attractor and the sensitivity on initial values are presented by numerical simulation, while feedback control method is used to eliminate chaos. Moreover, in the stage of periodic bifurcation outside the stability region, the increase of the adjustment speed of the leader could be incentive for choosing chaos. While in the chaos stage, the average profits of three firms are uncertain, which shows that the relative benefit is closely related to adjustment speed of bounded rationality.

Keywords Stackelberg model · Multi-period triopoly game · Chaos · Bounded rationality

JEL Classification C73 · D21 · D43 · D52 · C62

Extended author information available on the last page of the article

1 Introduction

An oligopoly market is a mixed market between perfect monopoly and monopolistic competition market. In such a typical market, only a few firms take the strategies of counterparts into account while focusing on consumer demands. Cournot proposed the first oligopoly model in 1838 (Cournot 1838), with two firms selling homogeneous products and competing with output. Bertrand model is a significant model of price competition (Bertrand 1883). As we all know, Nash equilibrium solution is a core content in classic game theory. A large amount of references have revolved around this solution concept. These equilibrium models provided some deterministic conclusions which were needed by economists. Given initial conditions, however, the status of system might still be unpredictable even after Lorenz discovered chaos behavior in deterministic system in 1963.

Chaos theory applied to dynamic oligopoly game model mainly combines traditional game theory with nonlinear dynamic system theory. This methodology relaxes the hypothesis of participants' complete rationality and complete information of market, and constructs a dynamic oligarchy game model given the condition of incomplete rationality. When the participants follow certain "behavior rules", it is not easy to maintain Nash equilibrium.

Puu's (1998) study revealed that equilibrium output is not consistent and periodic behavior exists. To make it more realistic, scholars put forward kinds of rationality expectations, which formed the cornerstone of this field. Modified Cournot models were further more proposed (Agiza 1998; Agiza et al. 1999; He and Li 2012). Even given 3 or 4 competitors, bifurcation and chaos attractors exist. Nonlinear demand function or cost function increased the complexion of dynamic game (Brianzoni et al. 2015; Askar et al. 2015). Cavalli and Naimzada (2015) verified that price elasticity of demand would cause a gradient adjustment in chaos. Besides expectations of bounded rationality and adaptive expectation, Tuinstra (2004) introduced local monopolistic approach (LMA). Based on LMA, oligopolists conjectured that the market demand function is linear and estimated equilibrium by solving a profit maximization problem (Bischi et al. 2007; Peng et al. 2016). With this approach, Nash equilibrium became unstable, and furthermore, flip bifurcation and Neimark–Sacker bifurcation occurred (Cavalli and Naimzada 2014; Bischi et al. 2015; Baiardi et al. 2015). Moreover, delayed dynamics based on bounded rationality assumptions attracts more attention. As delay parameter increases, the stability region would be enlarged (Ma and Wu 2013; Ding et al. 2014; Gori et al. 2015).

While the previous researches are based on homogeneous products, differentiated products become a hot topic for some scholars. The effect of product differentiation degree on chaos was verified (Agliari et al. 2016; Elsadany 2017). Yu and Yu's (2014) conclusion indicated that the greater the degree of product horizontal differentiation is, the more stable the Nash equilibrium of the system is. The studies of Fanti and Gori (2012), Askar and Alshamrani's (2014), revealed that a higher degree of product differentiation may destabilize the Nash equilibrium. Research of Shi et al. (2015) constructed a model with differentiated products which were produced by an original manufacture, and analyzed the effects of consumer's willingness-to-pay (WTP) on the stability of Nash equilibrium.

To some extent, chaos is a state that far away from equilibrium, and it means output or price is unpredictable. The situation in the market is difficult to control and oligarchs can not make accurate decisions. Thus measures should be taken to delay and eliminate chaos. Zhang and Ma (2012), Elsadany et al. (2013), as well as Yi and Zeng (2015) used delay feedback control in a dynamic game of four oligarchs. Du et al. (2013), Ma et al. (2014) stabilized chaos ultimately by adding appropriate limiter.

Generally speaking, most previous researches could be classified as follows: homogeneous products and differentiated products, in the perspective of product characteristics; price competition and output competition, in the oligarchs' decision-making variable perspective. However, no matter the model is Cournot, Bertrand or Hotelling model, or no matter the product is homogeneous, horizontally or vertically differentiated, the underlying model is still a static model with participants' simultaneous action. However, in realistic market, enterprises have different scales and power. Firms' actions are not well synchronized either. There're leader firms and followers. Stackelberg model is such a dynamic model that describes two players with sequential actions (von Stackelberg 1938). There are few literatures which apply complex oligopoly dynamics theory in Stackelberg model. Li and Ma (2016) studied the supply chains with probability selling using dynamics methodology. Peng and Lu (2015) investigated the effect of cost coefficient on equilibrium output according to dynamic system in a Stackelberg model. In their paper, the dynamic progress did not reflect this dynamic adjustment. In every period, the follower simplified the decision-making variable of the leader in stage 1 as a constant. However, in fact, in Stackelberg model, the follower could observe the leader's action in stage 2.

In our paper, we revise this adjustment progress and analyze a multi-period Stackelberg model with three oligarchs, one leader firm and two followers. In stage 1, the leader acts first; in stage 2, the two followers simultaneously adapt their outputs to the leader's output. The equilibrium outputs are solved by backward induction. A three-dimensional stability region is drawn. We observe periodic phenomena and chaotic behaviors with parameters exceeding the boundary of stability region. In our model, the decision-making variable of the leader in stage 1 is observed by two followers in every period.

This paper is organized as follows. In Sect. 2, a multi-period Stackelberg triopoly game and the locally asymptotic stability of equilibrium are investigated. In Sect. 3, some abundant simulation results on three-dimensional stability region, strange attractor and sensitivity on initial value are presented by value assignment. In Sect. 4, some conclusions are summarized through theoretical analysis and numerical simulation results.

2 The Model

2.1 Traditional Stackelberg Triopoly Game

Consider a traditional Stackelberg triopoly game with output competition. Demand function of the market is $Q=a-p$, where p is price of the market, a is price ceiling. $Q=q_1+q_2+q_3$ is supply of the market. For simplicity, we assume that every firm has the same linear cost function $c_i(q_i)=cq_i(i=1, 2, 3)$. Since a is price ceiling of the market, we have the reasonable assumption $a - c \triangleq \theta > 0$. The model has two stages. In the first stage, the leader firm 1 chooses the quantity q_1 and maximizes profit function $\pi_1 = (p - c)q_1$. In the second stage, the followers firms 2 and 3 observe q_1, and choose the quantity q_2 and q_3 to maximize the profit function respectively, namely to find a solution to the following maximum problem

$$\max_{q_i \geq 0} \pi_i = (p - c)q_i = (\theta - q_{-i} - q_i)q_i, \quad i = 2, 3$$

where q_{-i} represents outputs of other firms except firm i.

Consider the solution of optimization with backward induction. The marginal profit of firm $i(i=2, 3)$ is

$$\partial\pi_i/\partial q_i = \theta - q_{-i} - 2q_i \tag{1}$$

From the first order condition, for the given q_1, the optimal response function of firm $i(i=2, 3)$ is

$$q_i = (\theta - q_{-i})/2$$

It is easy to see $(\theta - q_{-i}) = a - c - q_{-i} \geq a - p - q_{-i} = \theta - q_{-i} \geq 0$. The intersection point of two response curves is

$$\begin{cases} q_2(q_1) = (\theta - q_1)/3 \\ q_3(q_1) = (\theta - q_1)/3 \end{cases} \tag{2}$$

Formula (2) is the optimal strategy of firms 2 and 3 for q_1.

In the first stage of game, firm 1 expects the outputs of firms 2 and 3 in formula (2) and maximizes its profit, by solving the maximization problem

$$\max_{q_1 \geq 0} \pi_1 = (\theta - q_{-1} - q_1)q_1$$

The marginal profit of firm 1 is

$$d\pi_1/dq_1 = (\theta - 2q_1)/3 \tag{3}$$

Let $d\pi_1/dq_1 = 0$, the Stackelberg equilibrium output is

$$q_1^s = \theta/2, \quad q_2^s = q_3^s = \theta/6 \tag{4}$$

2.2 Multi-period Stackelberg Tripoly Game and the Stability of Equilibrium

In the traditional Stackelberg model in Sect. 2.1, there is a hypothesis that every firm has complete information of the market, namely the three firms' cost function and demand function are known to all. The firms have the required rationality: in the first stage, firm 1 is able to predict the optimal strategies of q_2 and q_3 for a given q_1; and in the second stage, firms 2 and 3 could maximize their own profit function by observing q_1. However, such hypothesis mentioned is not realistic. For firms in the game, a more realistic assumption is that they act with regard to their limited market experience and other firms' behaviors.

Based on the traditional Stackelberg triopoly game proposed in Sect. 2.1, there are three hypotheses in the multi-period Stackelberg tripoly game in this section: firstly, firms are subjected to bounded rationality and limited information about market; secondly, every period includes multi-period Stackelberg game of two stages in Sect. 2.1; thirdly, firm 1 is able to estimate marginal profit in formula (3) with one unit change of q_1 according to its previous market experience. Firms 2 and 3 could estimate marginal profits in formula (1) after observing q_1.

Participants with bounded rationality adjust output $q_i(t+1)$ of next period according to current output $q_i(t)$ ($i=1, 2, 3$) and estimation of marginal profit. We have output adjustment dynamic equations as follows:

$$\begin{cases} q_1' = q_1 + v_1 q_1 \partial\pi_1/\partial q_1(q_1) \\ q_2' = q_2 + v_2 q_2 \partial\pi_2/\partial q_2(q_1', q_2, q_3) \\ q_3' = q_3 + v_3 q_3 \partial\pi_3/\partial q_3(q_1', q_2, q_3) \end{cases}$$

where $qi' = q_i(t+1)$, $q_i = q_i(t)$, $v_i > 0$ represents output adjustment speed of firm $i(i=1, 2, 3)$.

Substituting formula (1) and (3) into the equations, more detailed dynamic adjustment equations can be obtained:

$$\begin{cases} q_1' = q_1 + v_1 q_1(\theta - 2q_1)/3 = 0 \\ q_2' = q_2 + v_2 q_2(\theta - q_1' - q_3 - 2q_2) = 0 \\ q_3' = q_3 + v_3 q_3(\theta - q_1' - q_2 - 2q_3) = 0 \end{cases} \tag{5}$$

$q_1{}'$ on the right side of formula (5) indicates that in Stackelberg game firms 2 and 3 could observe current output of firm 1 in every period.

By such gradual adjustment, the final economic status depends on the fixed point of the system (5). To solve the fixed point, let $q_i{}' = q_i(i=1, 2, 3)$. The following algebraic equations is obtained

$$\begin{cases} v_1 q_1(\theta - 2q_1)/3 = 0 \\ v_2 q_2(\theta - q_1 - q_3 - 2q_2) = 0 \\ v_3 q_3(\theta - q_1 - q_2 - 2q_3) = 0 \end{cases}$$

Eight fixed points are $E_0 = (0, 0, 0)$, $E_1 = (0, 0, \theta/2)$, $E_2 = (0, \theta/2, 0)$, $E_3 = (\theta/2, 0, 0)$, $E_4 = (0, \theta/3, \theta/3)$, $E_5 = (\theta/2, 0, \theta/4)$, $E_6 = (\theta/2, \theta/4, 0)$, $E_7 = (\theta/2, \theta/6, \theta/6)$.

Where E_0 represents that every firm has no output; E_1, E_2 and E_3 represent monopoly firms 1, 2 and 3, respectively. E_4 represents that firms 2 and 3 compete in a Cournot game without firm 1, with equilibrium outputs of $\theta/3$ and $\theta/3$, respectively. E_5 and E_6 represent that firms 1 and 3, firms 1 and 2 compete in a Stackelberg game; with equilibrium outputs of $\theta/2$ and $\theta/4$, respectively. E_7 represents that the three firms compete in a Stackelberg game, with equilibrium outputs of $\theta/2$, $\theta/6$ and $\theta/6$, respectively. $E_0,\ldots, E_7$ are called boundary equilibria.

To discuss the locally asymptotic stability of fixed point $E_i(i=0, 1,\ldots,7)$, consider the Jacobian matrix of system (5):

$$J = \begin{bmatrix} 1 + v_1(\theta - 4q_1)/3 & 0 & 0 \\ -v_2q_2(1 + v_1(\theta - 4q_1)/3 & 1 + v_2(\theta - q_1' - q_3 - 4q_2) & -v_2q_2 \\ -v_3q_3(1 + v_1(\theta - 4q_1)/3 & -v_3q_3 & 1 + v_3(\theta - q_1' - q_2 - 4q_3) \end{bmatrix} \tag{6}$$

when $q_1 = 0$, J has a characteristic root $\lambda = 1 + v_1\theta/3 > 1$, so the fixed points of E_0, E_1, E_2, E_4 are not stable.

For $E_3 = (\theta/2, 0, 0)$,

$$J(E_3) = \begin{bmatrix} 1 - v_1\theta/3 & 0 & 0 \\ * & 1 + v_2\theta/2 & 0 \\ * & * & * \end{bmatrix}$$

where * means the characteristic roots of $J(E_3)$ is unrelated to this element.

The characteristic root of $J(E_3)$ is $\lambda = 1 + v_2\theta/2 > 1$, so E_3 is locally asymptotic instability.

For $E_5 = (\theta/2, 0, \theta/4)$,

$$J(E_5) = \begin{bmatrix} 1 - v_1\theta/3 & 0 & 0 \\ 0 & 1 + v_2\theta/4 & 0 \\ * & * & * \end{bmatrix}$$

The characteristic root of $J(E_5)$ is $\lambda = 1 + v_2\theta/4 > 1$, so E_5 is locally asymptotic instability.

For $E_6 = (\theta/2, \theta/4, 0)$,

$$J(E_6) = \begin{bmatrix} 1 - v_1\theta/3 & * & 0 \\ * & * & * \\ 0 & 0 & 1 + v_3\theta/4 \end{bmatrix}$$

The characteristic root of $J(E_6)$ is $\lambda = 1 + v_3\theta/4 > 1$, so E_6 is locally asymptotic instability.

By analyzing above, we get a proposition as follows.

Proposition *Boundary equilibria E_0, E_1, …, E_6 are not locally asymptotic stability.*

Next we investigate the stability of Stackelberg equilibrium E_7. We substitute formula (4) *into Jacobian matrix* (6):

$$J(E_7) = \begin{bmatrix} 1 - v_1\theta/3 & 0 & 0 \\ -v_2\theta(3 - v_1\theta)/18 & 1 - v_2\theta/3 & -v_2\theta/6 \\ -v_3\theta(3 - v_1\theta)/18 & -v_3\theta/6 & 1 - v_3\theta/3 \end{bmatrix}$$

The characteristic polynomial of $J(E_7)$ *is*

$$f(\lambda) = (\lambda - 1 + v_1\theta/3)g(\lambda)$$

where $g(\lambda)$ *is the characteristic polynomial of submatrix* $\begin{pmatrix} 1 - v_2\theta/3 & -v_2\theta/6 \\ -v_3\theta/6 & 1 - v_3\theta/3 \end{pmatrix}$, *that is*

$$\begin{aligned} g(\lambda) &= \begin{vmatrix} \lambda - 1 + v_2\theta/3 & v_2\theta/6 \\ v_3\theta/6 & \lambda - 1 + v_3\theta/3 \end{vmatrix} = \begin{vmatrix} \mu + v_2\theta/3 & v_2\theta/6 \\ v_3\theta/6 & \mu + v_3\theta/3 \end{vmatrix} \\ &= (\mu + v_2\theta/3)(\mu + v_3\theta/3) - (\theta/6)^2 v_2 v_3 \\ &= \mu^2 + \theta(v_2 + v_3)/3\mu + (\theta^2/12)v_2 v_3 \end{aligned}$$

where $\mu = \lambda - 1$.

Thus $f(\lambda) = (\mu + v_1\theta/3)g(\lambda) = 0$ *has three roots* $\mu_1 = -v_1\theta/3$, $\mu_2 = -(v_2 + v_3 + \sqrt{(v_2 - v_3)^2 + v_2 v_3})\theta/6$, $\mu_3 = -(v_2 + v_3 - \sqrt{(v_2 - v_3)^2 + v_2 v_3})\theta/6$.

Because a sufficient condition of locally asymptotic stability of E_7 is that all characteristic roots satisfy $-1 < \lambda < 1$, which equals to $-2 < \mu < 0$, $(\lambda = \mu + 1)$. Since $\mu_2 < \mu_3$, the sufficient condition of locally asymptotic stability of E_7 is

$$\mu_1 = -v_1\theta/3 > -2 \tag{7}$$

$$\mu_2 = -(v_2 + v_3 + \sqrt{(v_2 - v_3)^2 + v_2 v_3})\theta/6 > -2 \tag{8}$$

$$\mu_3 = -(v_2 + v_3 - \sqrt{(v_2 - v_3)^2 + v_2 v_3})\theta/6 < 0 \tag{9}$$

Formula (7) equals to $v_1 < 6/\theta$. Formula (9) equals to $v_2 + v_3 > \sqrt{(v_2 - v_3)^2 + v_2 v_3}$. Square ends of formula (9):

$$v_2^2 + v_3^2 + 2v_2 v_3 > (v_2 - v_3)^2 + v_2 v_3 = v_2^2 + v_3^2 - v_2 v_3$$

The last inequality is clearly established. So formula (9) is established.

Formula (8) equals to $v_2 + v_3 + \sqrt{(v_2 - v_3)^2 + v_2 v_3} < 12/\theta$. That is

$$\sqrt{(v_2 - v_3)^2 + v_2 v_3} < 12/\theta - (v_2 + v_3)$$

Square ends of the formula above:

$$v_2 + v_3 - \theta v_2 v_3/8 < 6/\theta$$

That is $v_3 < (6 - v_2\theta)8/\theta(8 - v_2\theta)$.

By analyzing above, we get a theorem as follows:

Theorem *The stability region of Stackelberg equilibrium* E_7 *is* $S = \{(v_1, v_2, v_3) | 0 \le v_1 < 6/\theta, \quad v_2 \ge 0, \quad 0 \le v_3 < (6 - v_2\theta)8/\theta(8 - v_2\theta)\}$.

The stability region of E_7 *is shown in Fig. 1.*

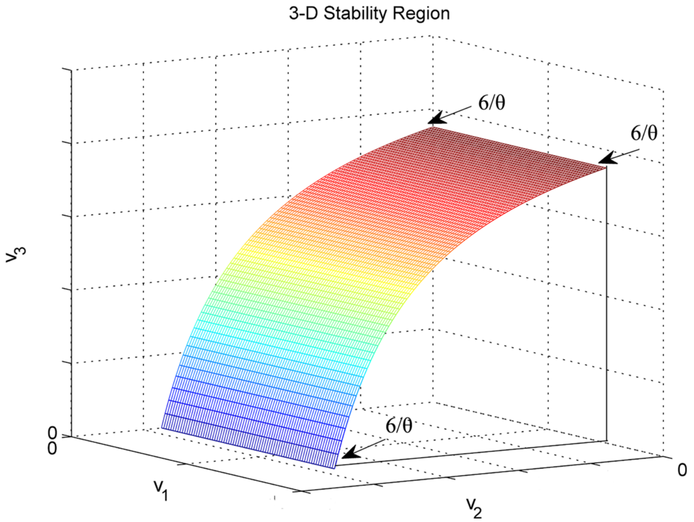

Fig. 1 Stability region of multi-period Stackelberg tripoly game equilibrium

The intersection point of S and v_1 axis is $(6/\theta, 0, 0)$; intersection point of S and v_2 axis is $(0, 6/\theta, 0)$; intersection point of S and v_3 axis is $(0, 0, 6/\theta)$.

In the multi-period Stackelberg game, traditional Stackelberg equilibrium outputs could be arrived by adjusting firms' outputs while keeping adjustment speed $(v_1, v_2, v_3) \in S$ using dynamic Eq. (5). When the adjustment speed is not in stability region S, we need the numerical simulation method to discuss the instability of E_7.

Besides, the effect of adjustment speed on the speed of converging to equilibrium is analyzed. Consider the nonlinear mapping $q' = \mathrm{T}(q)$ which is given by the system (5), where $q' = (q_1', q_2', q_3')$, $q = (q_1, q_2, q_3)$. q^* is the Stackelberg equilibrium. From the Taylor expansion:

$\mathrm{T}(q) = \mathrm{T}(q^*) + J_{q*} \cdot (q - q^*) + \mathrm{O}(\|q - q^*\|^2)$, where J_{q*} is the Jacobian matrix of mapping T at q^*. $|\mathrm{O}(\|q - q^*\|^2)| \leq \mathrm{c}\|q - q^*\|^2, c > 0$. Since $q^*$is a fixed point of T, for any q in the neighborhood of q^*, $q' = \mathrm{T}(q) \approx q^* + J_{q*} \cdot (q - q^*)$. Let $\mathrm{X} = q - q^*$, then X' $\approx J_{q*} \cdot \mathrm{X}$. Thus in the neighborhood of q^*, substitute the linear mapping $q' = J_{q*} \cdot \mathrm{X}$ for the nonlinear mapping $q' = \mathrm{T(x)}$ approximately. In this neighborhood, the speed of q converging to q^* depends on the speed of X converging to 0.

Let $\lambda_1, \lambda_2, \lambda_3 (\lambda_1 \neq \lambda_2 \neq \lambda_3)$ are three characteristic roots of J_{q*}. They are in the unit circle. λ_3 is a characteristic root of largest absolute value, that is $|\lambda_3| = \max\{|\lambda_1|, |\lambda_2|, |\lambda_3|\}$. It is easy to know the characteristic vectors a^1, a^2, a^3 corresponding to $\lambda_1, \lambda_2, \lambda_3$ respectively are linearly independent. For any $\mathrm{x} \in \mathrm{R}^3$, $\mathrm{X} = x_1 a^1 + x_2 a^2 + x_3 a^3$. Where $x_3 \neq 0$ is assumed. For a k linear mapping of X $J_{q*}^k \mathrm{X} = x_1 \lambda_1^k a^1 + x_2 \lambda_2^k a^2 + x_3 \lambda_3^k a^3$, we divide both ends by λ_3^k, $\frac{J_{q*}^k \mathrm{X}}{\lambda_3^k} = x_1 (\lambda_1/\lambda_3)^k a^1 + x_2 (\lambda_2/\lambda_3)^k a^2 + x_3 a^3$. We have $\lim\limits_{k \to \infty} \frac{J_{q*}^k \mathrm{X}}{\lambda_3^k} = x_3 a^3$, which illustrates that under the assumption of $x_3 \neq 0$, when $k \to \infty$, the speed of $J_{q*}^k \mathrm{X}$ converging to 0 and that of

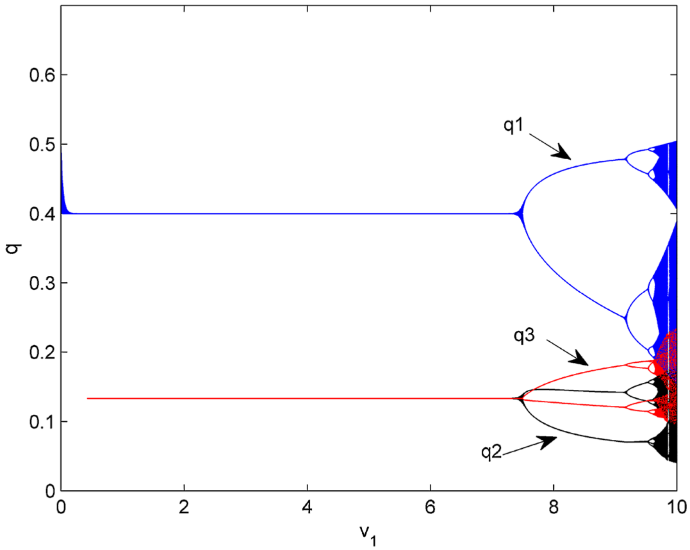

Fig. 2 Output bifurcation diagram with respect to v_1

λ_3^k converging to 0 have the same order. Therefore, under the assumption of nonlinear mapping, for any q in a small neighborhood of q^*, the speed of q converging to q^* approximate to that of λ_3^k converging to 0.

From the Theorems 13.5.3 and 13.5.4 (Robinson 2013), the largest Lyapunov exponent of q which is in the small neighborhood of q^* equals to $\ln|\lambda|$. Where λ is the characteristic root of largest absolute value of J_{q*}. Since λ is in the unit circle, the smaller the largest Lyapunov exponent is, the faster the speed of q_0 converging to q^* is; the greater the largest Lyapunov exponent is, the slower the speed of q_0 converging to q^* is.

3 Numerical Simulation

In this section, we present some complicated dynamic phenomena by numerical simulation. We set parameters as follows: $\theta = 0.8$, $v_2 = 3.5$, $v_3 = 3$. From the stability region of Stackelberg tripoly game equilibrium in Fig. 1, $v_1 = 7.5$ is the boundary point which corresponds to the first bifurcation point in Fig. 2. From Fig. 2, equilibrium outputs $q_1 = 0.4$, $q_2 = q_3 = 0.1333$ which illustrates the first-move advantage in Stackelberg model.

The largest Lyapunov exponent with respect to v_1 is drawn in Fig. 3. The first, second and third bifurcation point in Fig. 2 corresponds to L_{11} (7.4940, -0.0119), L_{12} (9.1280, 0.0011) and L_{13} (9.3280, 0.0011) in Fig. 3, respectively. Before chaos occurs, the largest Lyapunov exponents are all less than zero except bifurcation point; after chaos occurs, the largest Lyapunov exponents are almost all greater than zero.

The profit bifurcation diagram with respect to v_1 is drawn in Fig. 4. The average profit diagram with respect to v_1 is drawn in Fig. 5. We use the average profit of enter-

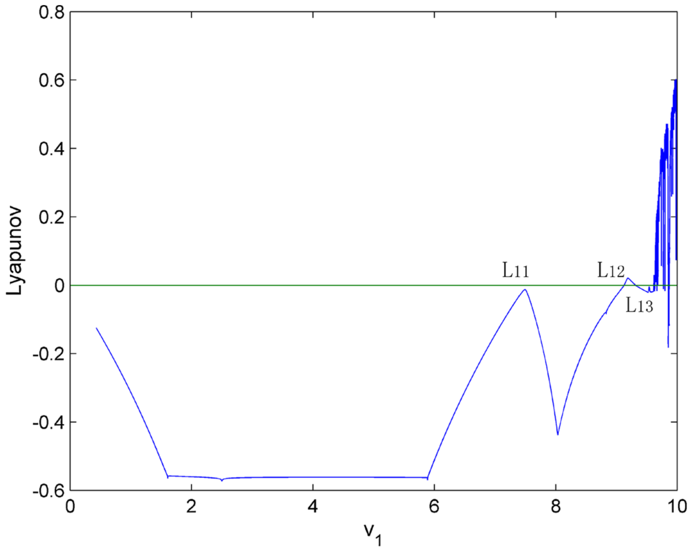

Fig. 3 Largest Lyapunov exponent diagram with respect to v_1

prises as the evaluation index to see if firms have the incentive for choosing chaos, and the effect of the adjustment parameter on profits. It is easy to see in the stability region, the average profits of three firms are equal to Stackelberg equilibrium profits. As v_1 increases, the adjustment parameter exceeds the stability region. Before chaos occurs, the average profit of firm 1 has an inverted U-shaped curve, which arrives maximum at $v_1 = 9.15$. The average profit of firm 2 presents an increase–decrease–increase path. The average profit of firm 3 is always increasing. The average profit of firm 1 is higher than that of firm 3, and the average profit of firm 3 is higher than that of firm 2, which illustrates that the leader firm still has classical game model's first-move advantage. The profit difference is caused by $v_2 > v_3$ entirely. Therefore, before chaos, firm 1 intends to adjust v_1 to $v_1 = 9.15$. The system is in stable bifurcation of 2 and 4 cycles at this time. When the adjustment parameter makes chaos happen, firms 1 and 3 may achieve higher average profits. However, if the parameter has a slight fluctuation, their profits will fluctuate dramatically. It is not a situation firms hope to see.

The strange attractors are shown from Figs. 6, 7, 8, and 9. We adjust v_1 while keeping $v_2 = 3.5$, $v_3 = 3$. The evolution progress of attractors could be observed as v_1 increases. We calculate the fractal dimension using three-dimensional system formula $d_L = 2 + \frac{\lambda_1+\lambda_2}{|\lambda_3|}$. From formulas (7), (8) and (9), $\lambda_3 = -1.6133$, $\lambda_2 = -0.3038$, $\lambda_1 = 0.5705$, $d_L = 2.165$, which implies that the system experiences chaos.

Next we investigate the sensitivity on initial values. From Figs. 10, 11, and 12, the subgraph 1 and subgraph 2 represent the orbits of q_1 with different initial values $(q_{10}, q_{20}, q_{30}) = (0.5, 0.2, 0.2)$ and $(q_{10}, q_{20}, q_{30}) = (0.501, 0.2, 0.2)$. We take three values of v_1: $v_1 = 5$, $v_1 = 9$, $v_1 = 9.8$ which represents equilibrium, bifurcation and chaos in the state of system respectively. As t increases, a tiny change $\Delta = 0.001$ of q_1 could cause three situations with different v_1. In Fig. 10, the two subgraphs are the same. It could not change the equilibrium of $q_1 = 0.4$. In Fig. 11, the two subgraphs are

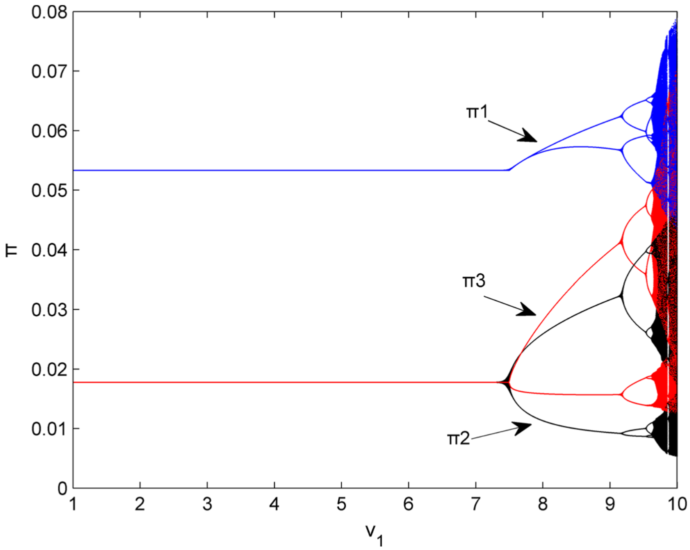

Fig. 4 Profit bifurcation diagram with respect to v_1

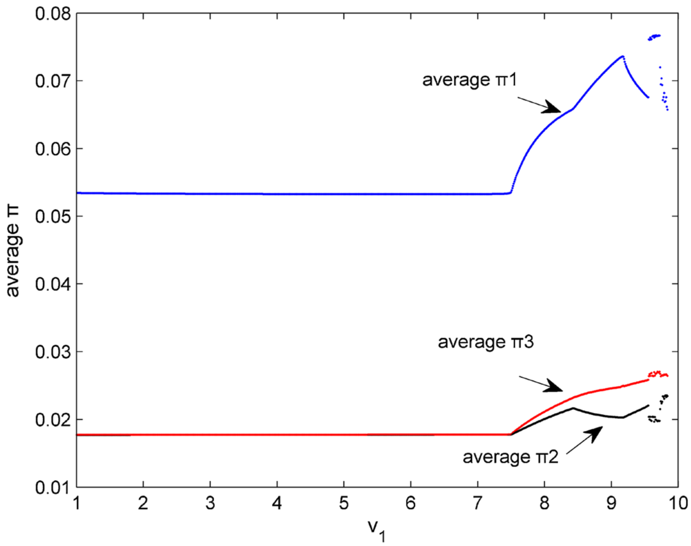

Fig. 5 Average profit diagram with respect to v_1

the same. The system is in periodic bifurcation. In Fig. 12, this subtle change causes a dramatic fluctuation of q_1. It shows that the system has a sensitive dependence on initial value of q_1 in chaos.

The subgraph 1 and subgraph 2 represent the orbits of q_2 with different initial values $(q_{10}, q_{20}, q_{30}) = (0.5, 0.2, 0.2)$ and $(q_{10}, q_{20}, q_{30}) = (0.5, 0.201, 0.2)$ in Figs. 13, 14 and 15. As t increases, a tiny change $\Delta = 0.001$ of q_2 could cause three situations

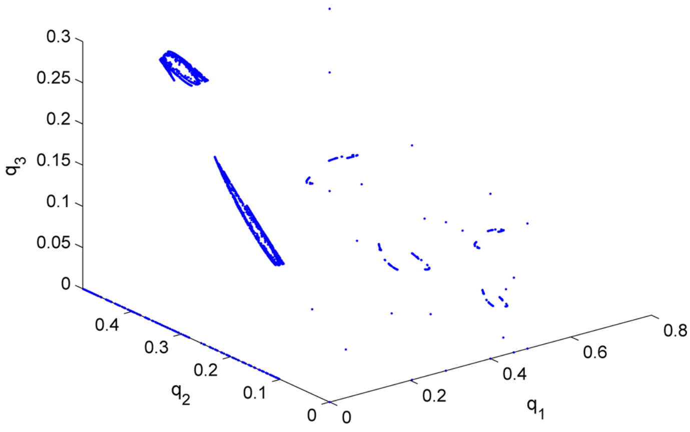

Fig. 6 Strange attractor for $v_1 = 9.5$, $v_2 = 3.5$, $v_3 = 3$

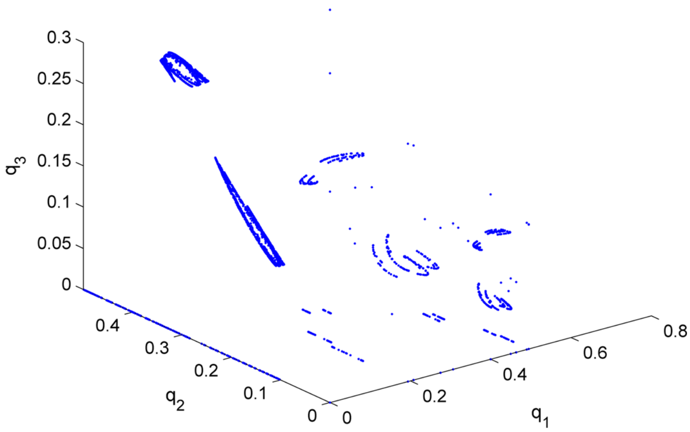

Fig. 7 Strange attractor for $v_1 = 9.6$, $v_2 = 3.5$, $v_3 = 3$

with different v_1. In Fig. 13, the two subgraphs are the same. It could not change the equilibrium of $q_2 = 0.1333$. In Fig. 14, the two subgraphs are the same. The system is in periodic bifurcation. In Fig. 15, this subtle change causes a dramatic fluctuation of q_2. It shows that the system has a sensitive dependence on initial value of q_2 in chaos.

The subgraph 1 and subgraph 2 represent the orbits of q_3 with different initial values $(q_{10}, q_{20}, q_{30}) = (0.5, 0.2, 0.2)$ and $(q_{10}, q_{20}, q_{30}) = (0.5, 0.2, 0.201)$ in Figs. 16, 17 and 18. As t increases, a tiny change $\Delta = 0.001$ of q_3 could cause three situations with different v_1. In Fig. 16, the two subgraphs are the same. It could not change the equilibrium of $q_3 = 0.1333$. In Fig. 17, the two subgraphs are the same. The system is in periodic bifurcation. In Fig. 18, this subtle change causes a dramatic fluctuation of q_3. It shows that the system has a sensitive dependence on initial value of q_3 in chaos.

In this paper, we consider the delayed feedback method to control system (5), which was proposed by Pyragas (1992, 1993). The main idea is to substitute partial

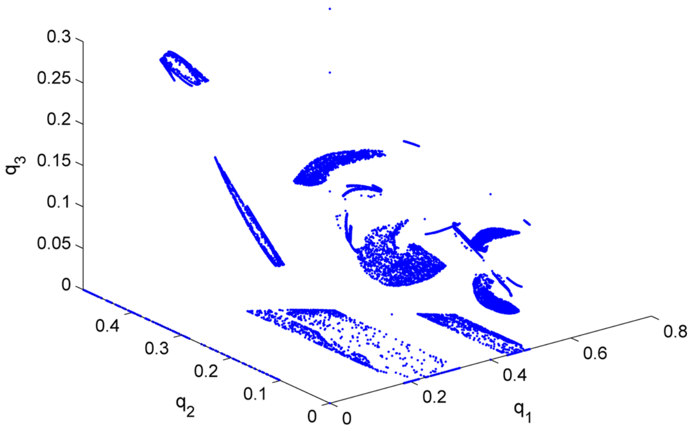

Fig. 8 Strange attractor for $v_1 = 9.7$, $v_2 = 3.5$, $v_3 = 3$

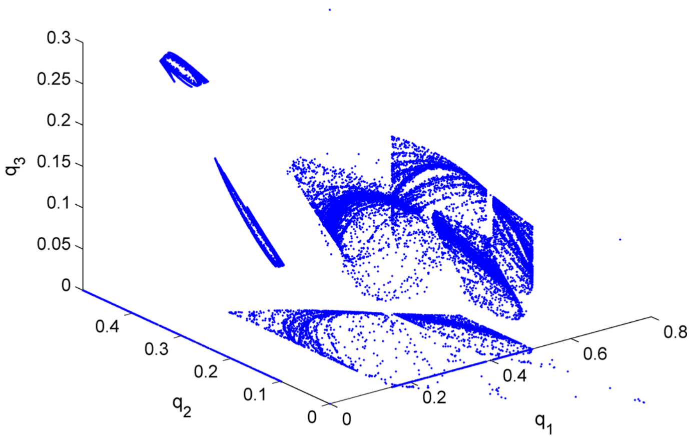

Fig. 9 Strange attractor for $v_1 = 9.8$, $v_2 = 3.5$, $v_3 = 3$

information of output signal of the system for output signal of the system, and return the system by time delay. A new control dynamic system is constructed as follows:

$$\begin{cases} q_1' = q_1 + v_1 q_1(\theta - 2q_1)/3(1+k) \\ q_2' = q_2 + v_2 q_2(\theta - q_1' - q_3 - 2q_2) \\ q_3' = q_3 + v_3 q_3(\theta - q_1' - q_2 - 2q_3) \end{cases} \tag{10}$$

The Jacobian matrix $J_c(E_7)$ of system (10) is

$$J_c(E_7) = \begin{bmatrix} 1 - v_1\theta/3(1+k) & 0 & 0 \\ -v_2\theta(3 - v_1\theta)/18 & 1 - v_2\theta/3 & -v_2\theta/6 \\ -v_3\theta(3 - v_1\theta)/18 & -v_3\theta/6 & 1 - v_3\theta/3 \end{bmatrix}$$

According to the characteristic equation of $J_c(E_7)$, and all characteristic roots locating in unit circle, the value range of control factor k could be obtained. The characteristic

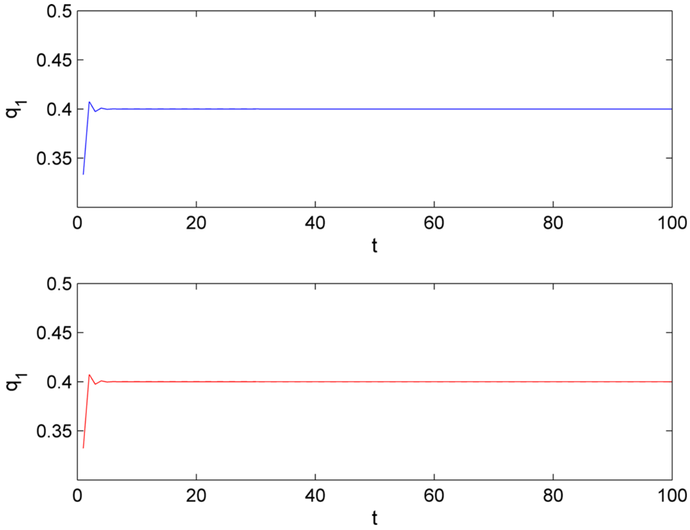

Fig. 10 Two orbits of q_1 with initial values $(q_{10}, q_{20}, q_{30}) = (0.5, 0.2, 0.2)$, $(q_{10}, q_{20}, q_{30}) = (0.501, 0.2, 0.2)$ and $v_1 = 5$

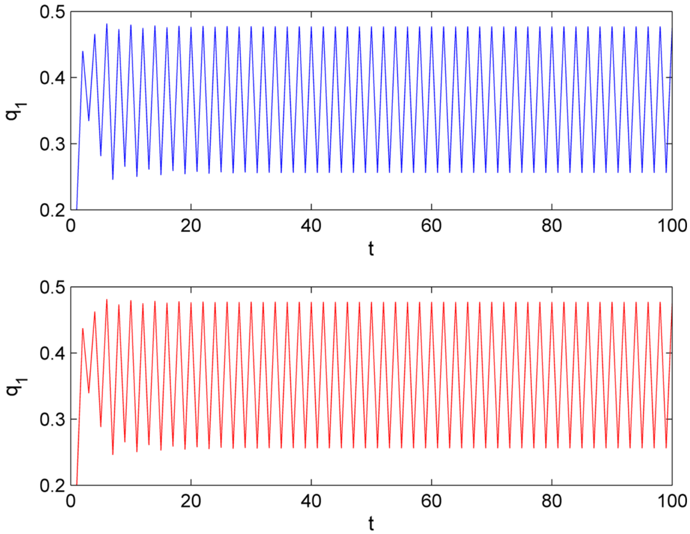

Fig. 11 Two orbits of q_1 with different initial values $(q_{10}, q_{20}, q_{30}) = (0.5, 0.2, 0.2)$, $(q_{10}, q_{20}, q_{30}) = (0.501, 0.2, 0.2)$ and $v_1 = 9$

roots of $J_c(E_7)$: $f_c(\lambda) = (\lambda - 1 + v_1\theta \big/ 3(1+k))g(\lambda)$, $g(\lambda)$ unchanged. From the proof process of Proposition, $k > \frac{v_1\theta}{6} - 1$. For $v_1 = 9.8$, $\theta = 0.8$, $k > 0.3067$.

Let k=0.4, the bifurcation diagram with respect to control factor k is shown in Fig. 19. The largest Lyapunov exponent with respect to k is drawn in Fig. 20. The first, second and third bifurcation point in Fig. 19 corresponds to L_{23} (0.3490, -0.0269), L_{22} (0.1140, -0.0024) and L_{21} (0.0520, 0.0024) in Fig. 20, respectively. Before equilib-

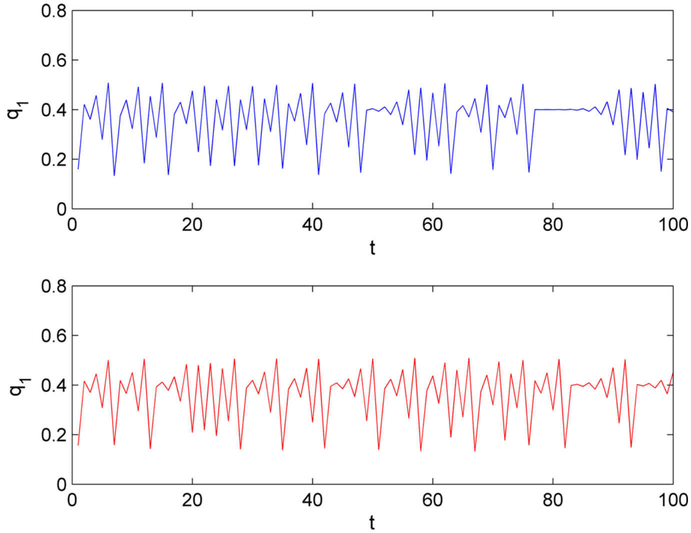

Fig. 12 Two orbits of q_1 with different initial values $(q_{10}, q_{20}, q_{30}) = (0.5, 0.2, 0.2)$, $(q_{10}, q_{20}, q_{30}) = (0.501, 0.2, 0.2)$ and $v_1 = 9.8$

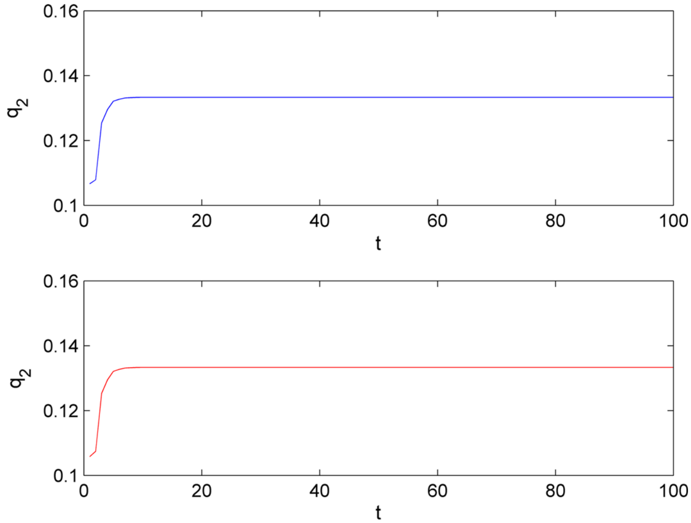

Fig. 13 Two orbits of q_2 with initial values $(q_{10}, q_{20}, q_{30}) = (0.5, 0.2, 0.2)$, $(q_{10}, q_{20}, q_{30}) = (0.5, 0.201, 0.2)$ and $v_1 = 5$

rium occurs, the largest Lyapunov exponents are all greater than zero except bifurcation point; after equilibrium occurs, the largest Lyapunov exponents are almost all less than zero.

The effects of control factor k on output and profit before and after chaos are drawn in Figs. 21 and 22. The outputs of manufactures finally stabilize at the equilibrium

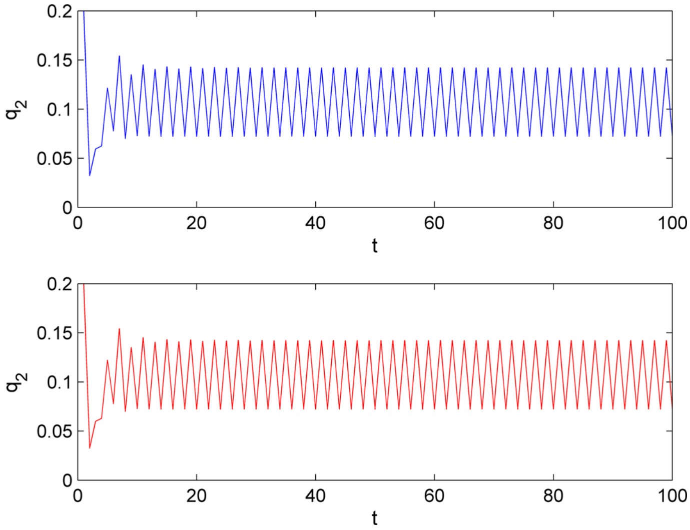

Fig. 14 Two orbits of q_2 with initial values $(q_{10}, q_{20}, q_{30}) = (0.5, 0.2, 0.2)$, $(q_{10}, q_{20}, q_{30}) = (0.5, 0.201, 0.2)$ and $v_1 = 9$

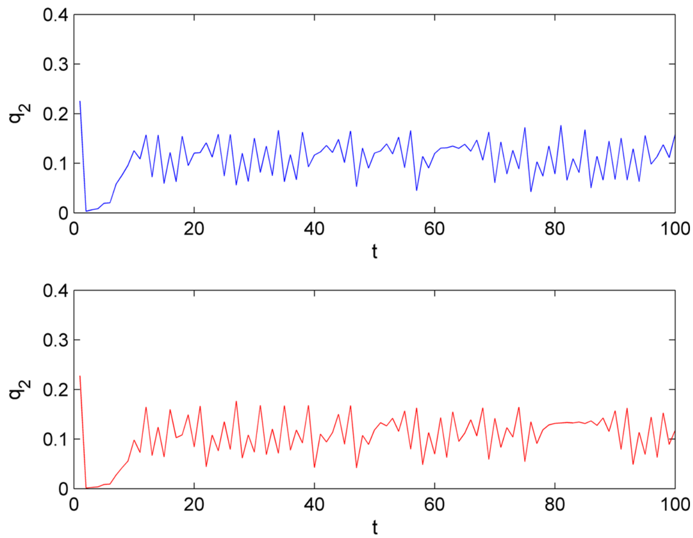

Fig. 15 Two orbits of q_2 with initial values $(q_{10}, q_{20}, q_{30}) = (0.5, 0.2, 0.2)$, $(q_{10}, q_{20}, q_{30}) = (0.5, 0.201, 0.2)$ and $v_1 = 9.8$

levels $q_1 = 0.4$, $q_2 = q_3 = 0.1333$. The profits of manufactures finally stabilize at the equilibrium levels $\pi_1 = 0.0534$, $\pi_2 = \pi_3 = 0.0178$.

The simulation results illustrate that in a multi-period triopoly Stackelberg model with sequential actions, with the increase of the speed of output adjustment, dynamic Stackelberg equilibrium becomes unstable and stays in chaos for a long time.

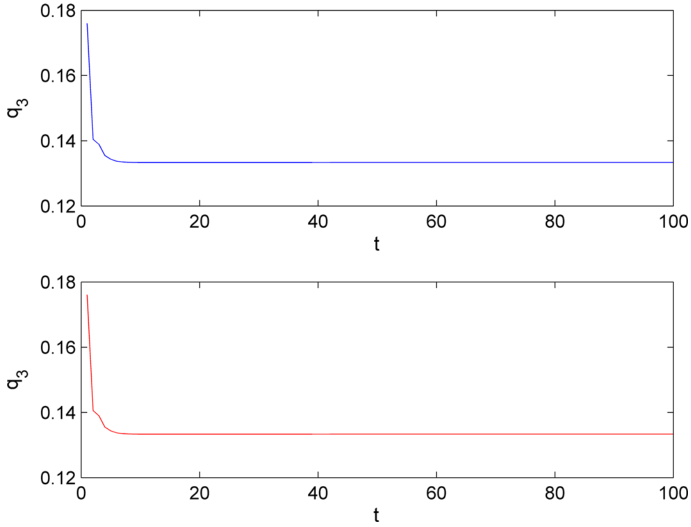

Fig. 16 Two orbits of q_3 with initial values $(q_{10}, q_{20}, q_{30}) = (0.5, 0.2, 0.2)$, $(q_{10}, q_{20}, q_{30}) = (0.5, 0.2, 0.201)$ and $v_1 = 5$

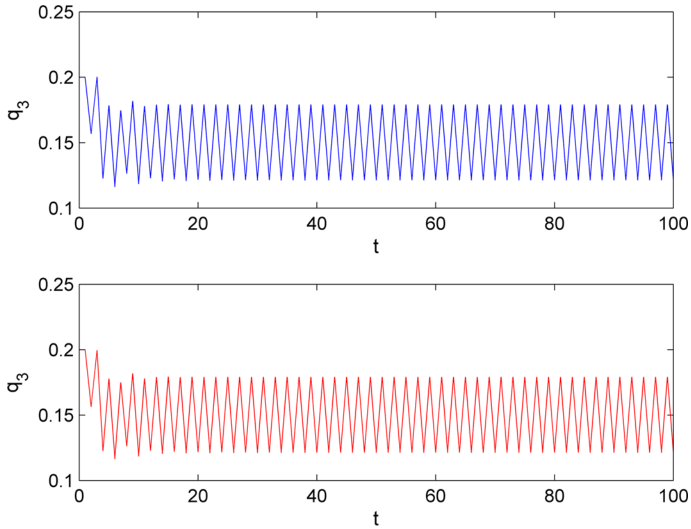

Fig. 17 Two orbits of q_3 with initial values $(q_{10}, q_{20}, q_{30}) = (0.5, 0.2, 0.2)$, $(q_{10}, q_{20}, q_{30}) = (0.5, 0.2, 0.201)$ and $v_1 = 9$

At last, we discuss the relation of largest Lyapunov exponent and iterative time by numerical simulation. According to Fig. 3, we divide the interval of v_1 into three intervals (0,1.61),[1.61,5.9],(5.9,7.5) and take two values in every small interval. We give the calculation data as Table 1 shows. As the largest Lyapunov exponents increases, the iterative time is increasing, the speed of convergence to equilibrium is slowing. The simulation verifies the theoretical result.

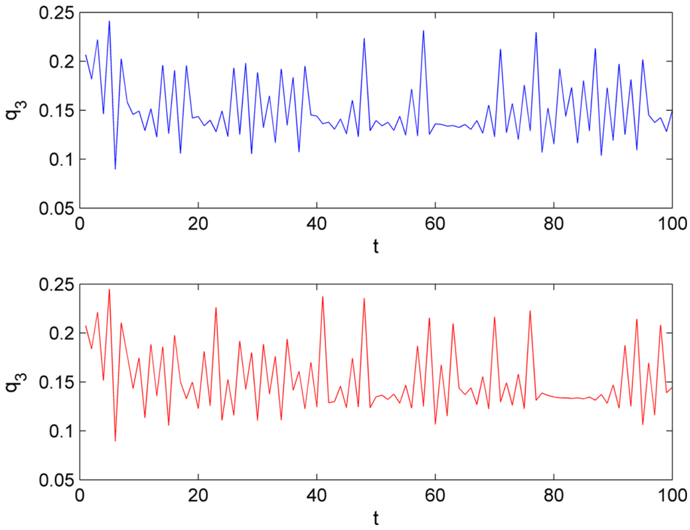

Fig. 18 Two orbits of q_3 with initial values $(q_{10}, q_{20}, q_{30}) = (0.5, 0.2, 0.2)$, $(q_{10}, q_{20}, q_{30}) = (0.5, 0.2, 0.201)$ and $v_1 = 9.8$

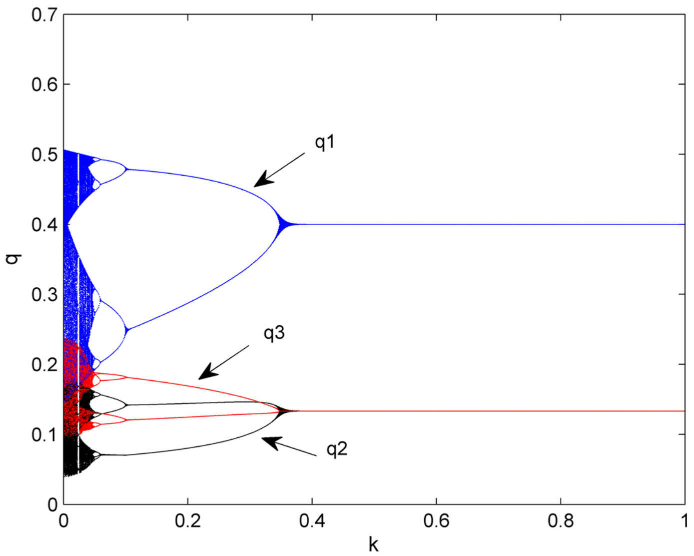

Fig. 19 Bifurcation diagram with respect to control factor k

4 Conclusion

Stackelberg model is a dynamic model which is different from Cournot, Bertand or Hotelling model. In this model, two players with different scales and power act sequentially. This game rule applies to the leader firm and the follower's strategy

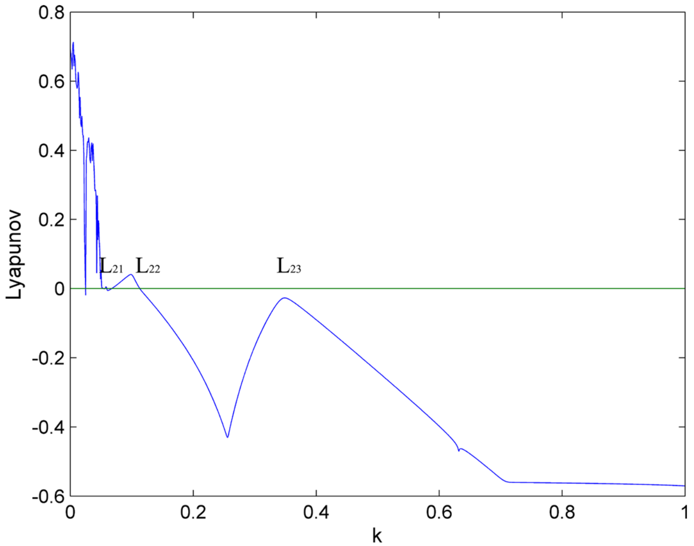

Fig. 20 Largest Lyapunov exponent diagram with respect to k

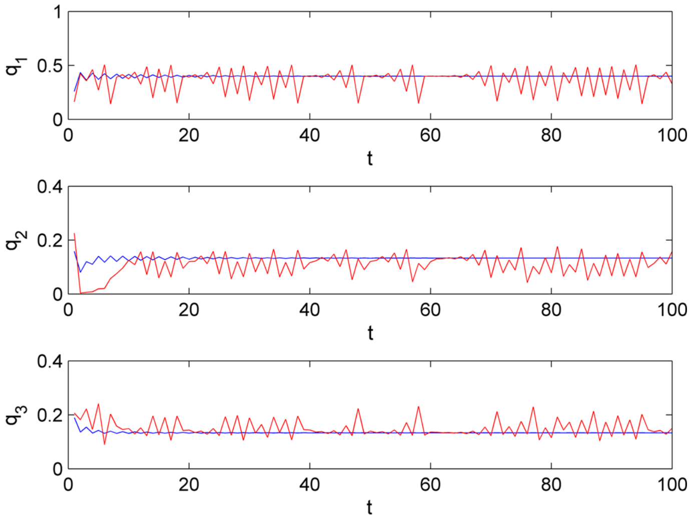

Fig. 21 Effects of control factor k on q_1, q_2, q_3 ($k = 0.4$)

behaviors in real market. It is a crucial model in traditional game theory, while there are few literatures which apply complex oligopoly dynamics theory in it.

The Stackelberg model has been modified in the paper. Our model improves dynamic adjustment equation of Peng and Lu (2015), which simplified the observing decision-making variable of the leader's actual output as a constant in every period. We construct a triopoly game model, with one leader firm and two followers who have bounded rationality and change the speed of output adjustment using marginal

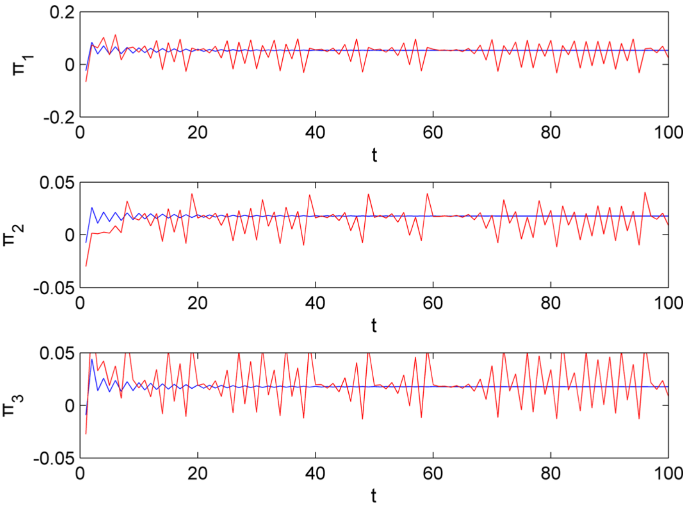

Fig. 22 Effects of control factor k on π_1, π_2, π_3 ($k=0.4$)

Table 1 The relation of largest Lyapunov exponent and iterative time

Value of v_1	Largest Lyapunov exponent	Iterative time n
0.65	-0.1778	38
1.05	-0.3319	22
3.157	-0.5621	10
5.6	-0.5621	10
6.4940	-0.3319	21
6.894	-0.1778	38

profit. The two followers' simultaneous move would take into account the effect of the leader's output on the demand in the market. In the equilibrium state, the outputs of the followers are one-third of the leader's.

A three-dimensional stability region is presented for the proposed multi-period Stackelberg equilibrium in this paper. The system will only be stable when the speed of output adjustment v_1, v_2 and v_3 are in such a three-dimensional stability region. The stability of fixed points in dynamic system is examined. Given fixed v_2 and v_3, period bifurcation and chaos occur as v_1 increases. The sensitivity on initial values and the strange attractor are demonstrated. Fractal dimension is calculated in chaos status. The leader's adjustment speed has influence on convergence to equilibrium. The greater the largest Lyapunov exponent is, the faster the speed of converging to equilibrium is. In addition, we choose the average profit to discuss whether the increase of adjustment speed of the leader is incentive for choosing chaos. The paths of three firms in stages of periodic bifurcation and chaos are different. However, the leader firm still remains first-move advantage outside the stability region. Finally, the system is restored equilibrium by delay feedback control method.

Acknowledgements This work was supported by Soft Science Research Project of Zhejiang Province (2016C35044). Thanks for the editors of Computational Economics and reviewers' work and consideration.

References

Agiza, H. N. (1998). Explicit stability zones for Cournot game with 3 and 4 competitors. *Chaos, Solitons and Fractals, 9*(12), 1955–1966.

Agiza, H. N., et al. (1999). Multistability in a dynamic Cournot game with three oligopolists. *Mathematics and Computers in Simulation, 51*(1), 63–90.

Agliari, A., et al. (2016). Nonlinear dynamics of a Cournot duopoly game with differentiated products. *Applied Mathematics and Computation, 281*(281), 1–15.

Askar, S. S., & Alshamrani, A. (2014). The dynamics of economic games based on product differentiation. *Journal of Computational and Applied Mathematics, 268*(1), 135–144.

Askar, S. S., et al. (2015). Dynamic Cournot duopoly games with nonlinear demand function. *Applied Mathematics and Computation, 259,* 427–437.

Baiardi, L. C., et al. (2015). Evolutionary competition between boundedly rational behavioral rules in oligopoly games. *Chaos, Solitons and Fractals, 79,* 204–225.

Bertrand, J. (1883). Theorie Mathematique de la Richesse Sociale. *Journal de Savants, 48,* 499–508.

Bischi, G. I., et al. (2007). Oligopoly games with local monopolistic approximation. *Journal of Economic Behavior and Organization, 62*(3), 371–388.

Bischi, G. I., et al. (2015). An evolutionary Cournot model with limited market knowledge. *Journal of Economic Behavior and Organization, 116,* 219–238.

Brianzoni, S., et al. (2015). Dynamics of a Bertrand duopoly with differentiated products and nonlinear costs: Analysis, comparisons and new evidences. *Chaos, Solitons and Fractals, 79,* 191–203.

Cavalli, F., & Naimzada, A. (2014). A Cournot duopoly game with heterogeneous players: Nonlinear dynamics of the gradient rule versus local monopolistic approach. *Applied Mathematics and Computation, 249,* 382–388.

Cavalli, F., & Naimzada, A. (2015). Effect of price elasticity of demand in monopolies with gradient adjustment. *Chaos, Solitons and Fractals, 76,* 47–55.

Cournot, A. (1838). *Recherches sur les Principes Mathematics de la Theorie de la Richesse*. Paris: Hachette.

Ding, Z., et al. (2014). Dynamical Cournot game with bounded rationality and time delay for marginal profit. *Mathematics and Computers in Simulation, 100,* 1–12.

Du, J., et al. (2013). Dynamics analysis and chaos control of a duopoly game with heterogeneous players and output limiter. *Economic Modelling, 33,* 507–516.

Elsadany, A. A. (2017). Dynamics of a Cournot duopoly game with bounded rationality based on relative profit maximization. *Applied Mathematics and Computation, 294,* 253–263.

Elsadany, A. A., et al. (2013). Complex dynamics and chaos control of heterogeneous quadropoly game. *Applied Mathematics and Computation, 219*(24), 11110–11118.

Fanti, L., & Gori, L. (2012). The dynamics of a differentiated duopoly with quantity competition. *Economic Modelling, 29*(2), 421–427.

Gori, L., et al. (2015). A continuous time Cournot duopoly with delays. *Chaos, Solitons and Fractals, 79,* 166–177.

He, X., & Li, K. (2012). Heterogeneous beliefs and adaptive behaviour in a continuous-time asset price model. *Journal of Economic Dynamics and Control, 36*(7), 973–987.

Li, Q., & Ma, J. (2016). Research on price Stackelberg game model with probabilistic selling based on complex system theory. *Communications in Nonlinear Science and Numerical Simulation, 30*(1), 387–400.

Ma, J., & Wu, K. (2013). Complex system and influence of delayed decision on the stability of a triopoly price game model. *Nonlinear Dynamics, 73*(3), 1741–1751.

Ma, J., et al. (2014). Complexity analysis of a master-slave oligopoly model and chaos control. *Abstract and Applied Analysis, 28*(4), 1–13.

Peng, Y., & Lu, Q. (2015). Complex dynamics analysis for a duopoly Stackelberg game model with bounded rationality. *Applied Mathematics and Computation, 271,* 259–268.

Peng, Y., et al. (2016). A dynamic Stackelberg duopoly model with different strategies. *Chaos, Solitons and Fractals, 85,* 128–134.

Puu, T. (1998). The chaotic duopolists revisited. *Journal of Economic Behavior and Organization, 33*(3), 385–394.
Pyragas, K. (1992). Continuous control of chaos by self-controlling feedback. *Physics Letters A, 170*(6), 421–428.
Pyragas, K. (1993). Experimental control of chaos by delayed self-controlling feedback. *Physics Letters A, 180*(1), 99–102.
Robinson, R. C. (2013). *An introduction to dynamical systems: Continuous and Discrete*. Washington, D.C.: American Mathematical Society.
Shi, L., et al. (2015). The dynamics of competition in remanufacturing: A stability analysis. *Economic Modelling, 50,* 245–253.
Tuinstra, J. (2004). A price adjustment process in a model of monopolistic competition. *International Game Theory Review, 6*(3), 417–442.
von Stackelberg, H. (1938). Probleme der unvollkommenen Konkurrenz. *Weltwirtschaftliches Archiv, 48,* 95–114.
Yi, Q. G., & Zeng, X. J. (2015). Complex dynamics and chaos control of duopoly Bertrand model in Chinese air-conditioning market. *Chaos, Solitons and Fractals, 76,* 231–237.
Yu, W., & Yu, Y. (2014). The complexion of dynamic duopoly game with horizontal differentiated products. *Economic Modelling, 41,* 289–297.
Zhang, J., & Ma, J. (2012). Research on the price game model for four oligarchs with different decision rules and its chaos control. *Nonlinear Dynamics, 70*(1), 323–334.

Affiliations

Yu Yu[1] · Weisheng Yu[2]

✉ Weisheng Yu
yws44@163.com

Yu Yu
yyuu_123@126.com

1 School of Economics, Zhejiang University of Finance and Economics, Hangzhou 310018, Zhejiang, People's Republic of China

2 Center for Quantitative Economics, Jilin University, Changchun 130000, Jilin, People's Republic of China