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Research paper

The stability of Bayesian Nash equilibrium of dynamic Cournot duopoly model with asymmetric information

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ABSTRACT

Few literatures apply complex oligopoly dynamics theory in games of incomplete information. This paper aims at analyzing dynamic behaviors of Bayesian game. A dynamic Cournot model with asymmetric information is proposed based on adaptive expectation and bounded rationality. Theoretical analysis draws two important conclusions: firstly, Bayesian Nash equilibrium of dynamic Cournot duopoly model with two players of adaptive expectation is always globally asymptotically stable. Secondly, Bayesian Nash equilibrium of dynamic Cournot duopoly model with players of adaptive expectation and gradient rule based on marginal profit is locally asymptotically stable only when parameters satisfy certain conditions. In our model, a firm of uncertain cost function is designed. A probability parameter θ of private type which differentiates high cost and low cost is introduced. Bifurcation, or even chaos with respect to θ, is performed by simulation which implies that large possibility of high-cost production yields easier chaos in duopoly market. High adjustment speeds of output form a three-dimensional strange attractors region. The unstable system's negative impact on equilibrium output and profit highlights the importance of system stability. Chaos control is in order to stabilize the equilibrium of the improved dynamic Cournot model with asymmetric information.

1. Introduction

An oligopoly is a market form in which a market or industry is dominated by a small number of sellers. It is more complex than monopoly and perfect competition. In such a typical market, a few firms control the market and provide the whole supply. Therefore oligopolists have to consider not only the effects of their own strategies on consumer demands, but also those of their opponents. According to players' order of action, traditional game models could be divided into static and dynamic game model. In static game models, Cournot model [9] and Bertrand model [6] are two famous models. Cournot model was the first oligopoly model. In Cournot model, two firms produce homogeneous products and compete with output. While sometimes firms make strategies of price. Bertrand model showed a significant model of price competition. If firms have different scales and power, the weaker firm may act after observing the action of the stronger firm. We call them the leader firm and the follower. Stackelberg model is such a dynamic model which is characterized by firms' sequential actions [25].

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In classic game theory, Nash equilibrium is an important concept. A large amount of references revolve around it. The game models provide deterministic conclusions using Nash equilibrium. However, Lorenz discovered chaos in deterministic system in 1963. Analysis of chaos in economic system attracts researchers' concern. Chaos theories applied to dynamic oligopoly game model mainly combines traditional game theory with chaos dynamics theory. This methodology, which relaxes the hypothesis of participants' complete rationality and complete information of market, has become an important tool to analyze dynamics behaviors of oligarchs. If players follow "behavior rules", Nash equilibrium would be unstable.

Firstly, scholars relax the assumption of player's complete rationality. To make the study more realistic, they put forward kinds of rationality expectations, which formed the cornerstone of this field, such as bounded rationality, adaptive expectation, naïve rationality, local monopolistic approach (LMA) and so on. Homogeneous products first attracted more concern. In Cournot model, the output of Nash equilibrium had a possibility of the periodic and more complex phenomenon [16,23]. Agiza et al. [1] discussed the stability conditions of the fixed points. In face of diseconomies of scale, bounded rational and adaptive duopolists were shown to experience chaos [11]. Yi and Zeng [28] demonstrated that the stability of Nash equilibrium strongly depended on the adjustment speed of bounded rationality player.

Other researchers assume that firms have nonlinear demand and cost functions to increase the complexity of dynamic game. Ahmed and Elettreby [4] introduced a multi-market Cournot model using Puu's [22] approach. Bischi et al. [7] considered a model with fixed fraction of firms in two complementary groups: Best Reply and LMA adjustment. Cournot-Bertrand mixed duopoly game offered a new research possibility. Two firms competing with price and output respectively came to some different conclusions. Ma and Pu [17] found that either the change of output modification speed or that of price modification speed would cause the market to the chaotic state. Andaluz et al. [5] pointed out that players' behavior rule determined the local stability of the Nash equilibrium, which was irrelevant to the competition type.

The effect of differentiated goods on chaos is also an interesting topic in this field. Yu and Yu [29] considered players' dynamic adjustment process in Hotelling model. A higher speed of price adjustment facilitated a periodic or even chaotic status. Ahmed et al. [3] derived the demand function from a CES utility function. Agliari et al. [2] and Brianzoni et al. [8] also improved a Cournot or Bertrand model to study a differentiated product model. They demonstrated that the differentiation degree of product destabilized the Nash equilibrium, while some researchers believed that the differentiation degree of product promoted the Nash equilibrium [14].

Besides static model of Cournot, Bertrand and Hotelling, dynamic model such as Stackelberg model also attracts researchers' concern. Peng and Lu [21] used backward induction to solve Stackelberg model and discussed the local stability of equilibrium. In Peng et al.'s [20] paper, they assumed that firms announced plan products sequentially in planning phase and acted simultaneously in production phase. The model parameters would drive chaos.

Researchers reach a consensus on the result that delay rationality enlarged stability region [10,12,15,18]. Because chaos means that the system is far away from equilibrium, effective measures should be taken to control. Yang et al. [27] limited chaos to periodic bifurcation by imposing constraints on difference equation. Ott et al. [19] first put forward a chaos control method, which is called OGY method. Pyragas [24] proposed a delayed feedback method most scholars focused on [13,26].

Complete information means each player understands payoff functions of others, and all payoff functions are common knowledge to everyone. Players are able to expect opponent's action and make the optimal response. In many economy and management issues, this hypothesis could not be reached. For instance, in Prisoner's Dilemma, player's opponent may be rational or irrational. In negotiations, the opponent may be tough or tolerant. In firms' output and price competition forms, cost function of the counterpart may be high or low. The private information makes opponents' payoff function uncertain. Accordingly study on game of incomplete information accords nearly with actual economic situation. In similar research articles, although considering bounded rationality and asymmetric information, whether Bayesian Nash equilibrium could be achieved by dynamic system is not demonstrated.

Although the above references revolve around incomplete rationality, incomplete information with bounded rationality is neglected. Therefore, relaxing players' complete information is a noteworthy issue. Combining players' incomplete information with incomplete rationality and chaotic theory is one of contributions this paper considers. Specifically, a dynamic Cournot duopoly model with asymmetric information is proposed. The uncertain cost function introduces a probability parameter of private type which gives some new message. The numerical simulation presents that large possibility of high-cost production yields easier chaos in duopoly market. The second innovation is that we prove that Bayesian Nash equilibrium could be achieved by the evolution of dynamic system. The third one is that the globally asymptotic stability condition of Bayesian Nash equilibrium is given. Moreover, chaos is controlled to stabilize the equilibrium by delayed feedback method.

The paper is arranged as follows. In Section 3, a static Cournot duopoly model with complete rationality and asymmetric information is investigated. The Bayesian Nash equilibrium is solved. In Section 4, a dynamic Cournot duopoly model with adaptive expectation and asymmetric information is constructed. In Section 5, a dynamic Cournot duopoly model with heterogeneous expectations and asymmetric information is formulated. The globally and locally asymptotic stability of Bayesian Nash equilibrium in the above sections are analyzed. In Section 6, some simulation results are presented. In Section 6, we draw some conclusions through theoretical analysis and numerical simulation.

2. Bayesian Nash equilibrium of static Cournot duopoly model with complete rationality and asymmetric information

Traditional game theory models could be divided into two types: games of complete and incomplete information. Bayesian Nash equilibrium is mainly used to discuss the equilibrium state in static incomplete (asymmetric) information game model. The definition is as follows. The payoff function of player i $(=1,\ 2,...n)$ is type-contingent: $u_i = u_i(q_{1,}q_2,\cdots q_n; c_{1,}c_2,\cdots c_n)$. q_i is player i's action. c_i is player i's private type. c_i is known to player i, while it is a random variable with a known probability distribution for other players (except player i). Player i's strategy is a mapping from the type space T_i to the action space A_i: $s_i(c_i) = q_i$. Where $c_i \in \mathrm{T}_i$, $q_i \in A_i$. The strategy combination $(s_i{}^*(\cdot), s_{-i}{}^*(\cdot))$ (where $s_{-i}{}^*(\cdot)=(s_1{}^*(\cdot),\cdots s_{i-1}{}^*(\cdot), s_{i+1}{}^*(\cdot),\cdots s_n{}^*(\cdot))$) is Bayesian Nash equilibrium of incomplete information static game. For a fixed $s^*_{-i}(\cdot)$, any $c_i \in \mathrm{T}_i, s_i{}^*(c_i)=q_i{}^*$, player i maximizes the conditional expectation payoff, that is $s_i{}^*(c_i)=\arg\max_{q_i} \mathrm{E}_{p_i}(u_1(s_i{}^*(t_{-i}), q_i; t_1, t_2,\cdots t_n))$. where p_i is player i's private type. Other players' type is marked as $c_{-i} = (c_1, ..., c_{i-1}, c_{i+1}, ...c_n)$. The conditional probability of player i is $p_i=p_i(c_{-i}|c_i)$. The concept of Bayesian Nash equilibrium is used to analyze a static duopoly Cournot model with complete rationality and asymmetric information in the following.

It is assumed that there are firm 1 and firm 2 in the market. They produce homogeneous products and compete with output. Their outputs are marked as q_1 and q_2. The inverse demand function of the market is $P(Q)=a$-Q, where $Q=q_1+q_2$ is the market supply. Firm 1 has a production cost function $C_1(q_1)=c_1{\cdot}q_1$, in which the marginal cost c_1 is known to each other. Firm 2 has a production cost function $C_2(q_2)=c_2{\cdot}q_2$. The marginal cost c_2, which is private information, could take two values $c_h = c_1 + \varepsilon,\ c_l = c_1 - \varepsilon (c_1 \succ 0,\ c_2 \succ 0,\ \varepsilon \succ 0)$, while the probability distribution $P(c_2 = c_h) = \theta,\ P(c_2 = c_l) = 1 - \theta(\theta \in (0, 1))$ is known to both firms. It is easy to know the mathematical expectation of c_2 is

$$Ec_2 = \theta(c_1 + \varepsilon) + (1 - \theta)(c_1 - \varepsilon) = c_1 + (2\theta - 1)\varepsilon$$

We have the hypothesis of parameters satisfying the condition:

$$a - c_1 \succ 2\varepsilon \tag{1}$$

To solve Bayesian Nash equilibrium of static Cournot duopoly model with incomplete information, we need to figure out the best reply function of two firms.

First of all, we consider the best reply function of firm 2. Given q_1 and the private marginal cost c_2, firm 2 chooses $q_2(c_2)$ to maximize his profit function namely solving the maximization problem:

$$\max_{q_2(c_2)} \pi_2 = (a - q_1 - q_2(c_2) - c_2)q_2(c_2)$$

His marginal profit is

$$\partial E\pi_2/\partial q_2 = a - q_1 - c_2 - 2q_2(c_2) \tag{2}$$

From the first order condition $\partial E\pi_2/\partial q_2 = 0$, the best reply function of firm 2 is

$$q_2(q_1; c_2) = \begin{cases} (a - c_2 - q_1)/2 & q_1 \le a - c_2 \\ 0 & q_1 \succ a - c_2 \end{cases} \qquad c_2 = c_h, c_l \tag{3}$$

Or

$$q_2(q_1; c_h) = \begin{cases} (a - c_h - q_1)/2 & q_1 \le a - c_h \\ 0 & q_1 \succ a - c_h \end{cases}$$

$$q_2(q_1; c_l) = \begin{cases} (a - c_l - q_1)/2 & q_1 \le a - c_l \\ 0 & q_1 \succ a - c_l \end{cases}$$

The mathematical expectation of the best reply function q_2 with respect to q_1 and c_2 is

$$\begin{aligned} Eq_2 &= \theta q_2(q_1; c_h) + (1 - \theta)q_2(q_1; c_l) \\ &= \begin{cases} (a - c_1 + (1 - 2\theta)\varepsilon - q_1)/2 & q_1 \le a - c_h \\ (1 - \theta)(a - c_1 + \varepsilon - q_1)/2 & a - c_h \prec q_1 \le a - c_l \\ 0 & q_1 \succ a - c_l \end{cases} \end{aligned} \tag{4}$$

Now consider the best reply function of firm 1.

Firm 1 could not obtain marginal cost c_2 of firm 2, but he knows its probability distribution. Thus firm 1 is able to maximize his mathematical expectation of profit only by determining q_1, i.e.,

$$\max_{q_1(c_2)} E\pi_1 = E(a - q_1 - q_2(c_2) - c_1)q_1 = (a - q_1 - Eq_2(c_2) - c_1)q_1$$

The marginal profit of firm 1 is

$$\partial E\pi_1/\partial q_1 = a - Eq_2(c_2) - c_1 - 2q_1 \tag{5}$$

From the first order condition $\partial E\pi_1/\partial q_1 = 0$, the best reply function of firm 1 is

$$q_1(Eq_2) = \begin{cases} (a - c_1 - Eq_2(c_2))/2 & Eq_2(c_2) \le a - c_1 \\ 0 & Eq_2(c_2) \succ a - c_1 \end{cases} \tag{6}$$

Where Eq_2 is given by formula (4).

We propose a set of hypotheses of complete information.

Firstly, firm 1 could expect the best reply function of firm 2 in formula (3), solve the mathematical expectation of best reply function in formula (4), and determine his best reply in formula (6).

Secondly, firm 2 could expect the best reply of firm 1 in formula (6) and determine his best reply in formula (3).

Based on the above hypotheses, from formula (3), (4) and (6), Bayesian Nash equilibrium could be calculated.

Proposition 1. *If parameters satisfy formula (1), firm 2's mathematical expectation of best reply function is* $Eq_2 = (a - c_1 - q_1 + (1 - 2\theta)\varepsilon)/2$.

Proof. From formula (4), if neither Eq_2=0 nor $Eq_2 = (1-\theta)(a - c_1 + \varepsilon - q_1)/2$ could establish, Proposition 1 is proved.

Firstly, if Eq_2=0, from formula (4) and (6), $q_1 \succ a - c_l = a - c_1 + \varepsilon$, $q_1 = (a - c_1)/2$. However, $(a - c_1)/2 \succ a - c_1 + \varepsilon$ could not establish. Thus Eq_2=0 could not establish either.

Secondly, if $Eq_2 = (1-\theta)(a - c_1 + \varepsilon - q_1)/2$, we have $Eq_2 \le a - c_1$. So from formula (6): $q_1 = (a - c_1 - (1-\theta)(a - c_1 + \varepsilon - q_1)/2)/2$. By calculation, $q_1 = ((1+\theta)(a - c_1) - (1-\theta)\varepsilon)/(3+\theta)$. However, under the assumption of formula (1), $q_1 \succ a - c_1 - \varepsilon$ could not establish.

Since $q_1 \succ a - c_1 - \varepsilon$ is equivalent to $((1+\theta)(a - c_1) - (1-\theta)\varepsilon)/(3+\theta) \succ a - c_1 - \varepsilon$, namely $(1+\theta)\varepsilon \succ a - c_1$. This is inconsistent with formula (1).

Thus $Eq_2 = (1-\theta)(a - c_1 + \varepsilon - q_1)/2$ is not possible either.

Finally, $Eq_2 = (a - c_1 - q_1 + (1 - 2\theta)\varepsilon)/2$. □

Proposition 2. *If parameters satisfy formula (1), Bayesian Nash equilibrium of static Cournot duopoly model with asymmetric information is:*

$$q_1{}^* = (a - c_1 + (2\theta - 1)\varepsilon)/3 \tag{7}$$

$$q_2{}^*(c_2) = \begin{cases} (a - c_1 - (1+\theta)\varepsilon)/3 & c_2 = c_h \\ (a - c_1 + (2-\theta)\varepsilon)/3 & c_2 = c_l \end{cases} \tag{8}$$

Proof. Firstly, if $Eq_2 \le a - c_1$, from formula (6) and (5), we have:

$$q_1 = (a - c_1 - Eq_2)/2 = (a - c_1 - (a - c_1 - q_1 + (1 - 2\theta)\varepsilon/2))/2$$

The solution of this equation is $q_1{}^* = (a - c_1 + (1 - 2\theta)\varepsilon)/3$, so formula (7) establishes.

From formula (3), when $q_1^* \le a - c_h$, $q_2^*(c_2) = \{\begin{matrix} (a - c_1 - (1+\theta)\varepsilon)/3 & c_2 = c_h \\ (a - c_1 + (2-\theta)\varepsilon)/3 & c_2 = c_l \end{matrix}$ therefore formula (8) establishes. But $Eq_2{}^*(c) \le a - c_1$ and $q_1{}^* \le a - c_h$ need to be verified.

Easy to know, $Eq_2{}^*(c) \le a - c_1$ is equivalent to $(1 - 2\theta)\varepsilon \le a - c_1$ and $q_1{}^* \le a - c_h$ is equivalent to $(1+\theta)\varepsilon \le a - c_1$. □

Thus on the hypothesis of formula (1), we have proved Proposition 2.

In this section, under the rationality assumption of Bayesian Nash equilibrium, a static Bayesian Nash equilibrium of Cournot model with asymmetric information is given. The equilibrium output of firm 1 is $q_1{}^* = (a - c_1 + (2\theta - 1)\varepsilon)/3$, which implies that the greater a or θ is, the greater the equilibrium output of firm 1 is; and the greater the marginal cost c_1 is, the smaller the equilibrium output of firm 1 is. The equilibrium strategy of firm 2 is

$$q_2{}^*(c_2) = \begin{cases} (a - c_1 - (1+\theta)\varepsilon)/3 & c_2 = c_h \\ (a - c_1 + (2-\theta)\varepsilon)/3 & c_2 = c_l \end{cases},$$

which is related to his private type, that is the marginal cost c_2. The greater a is, the greater the equilibrium output of firm 2 is. The greater c_1 or θ is, the smaller the equilibrium output of firm 2 is. In particular, $q_2{}^*(c_l) > q_2{}^*(c_h)$.

3. The stability of Bayesian Nash equilibrium of dynamic Cournot duopoly model with adaptive expectation and asymmetric information

In most situations, firms could not have the rationality which is assumed in Section 3, and they may not have complete information about market demand and production cost functions. Therefore, there are two problems. Firstly, could Bayesian Nash equilibrium be achieved? Secondly, is Bayesian Nash equilibrium stable? To answer the questions, we improve the Cournot duopoly model in Section 3 and give the dynamic game model. We assume that the probabilities that firm 1 meets firm 2 of high cost and low cost in the market are θ and 1–θ respectively in every period t (=1, 2, 3, …), and every firm adjusts his own output with adaptive expectation rule, namely firms adjusting output of next period according to his own output of previous period and an estimation of the best reply. Then the dynamic adjustment equations are:

$$\begin{cases} q_1{}' = \alpha_1 q_1 + (1-\alpha_1)(a - c_1 - \theta q_{2h} - (1-\theta) q_{2l})/2 \\ q_{2h}{}' = \alpha_2 q_{2h} + (1-\alpha_2)(a - c_h - q_1)/2 \\ q_{2l}{}' = \alpha_2 q_{2l} + (1-\alpha_2)(a - c_l - q_1)/2 \end{cases} \tag{9}$$

Where q_i' is firm i's (i = 1, 2 h, 2l) output of next period, and q_i is firm i's output of current period. $q_{2h} = q_2(c_h), q_{2l} = q_2(c_l)$. Easy to know, Bayesian Nash equilibrium achieved from formula (7) and (8) is the unique fixed point of formula (9). We could prove the following proposition:

Proposition 3. *On the hypothesis of adaptive expectation, Bayesian Nash equilibrium of dynamic Cournot duopoly model with asymmetric information is always globally asymptotically stable.*

Proof. To illustrate the stability of Bayesian Nash equilibrium, the dynamic equations (9) is written as

$$\mathrm{X}(t+1) = \mathrm{J}\mathrm{X}(t) + b \quad t = 0, 1, 2 \cdots$$

Where

$$\mathrm{X}(t) = (q_1(t), q_{2h}(t), q_{2l}(t))^{\mathrm{T}},$$

$$b = ((1-\alpha_1)(a-c_1)/2,\ (1-\alpha_2)(a-c_1-\varepsilon)/2,\ (1-\alpha_2)(a-c_1+\varepsilon)/2))^{\mathrm{T}},$$

$$J = \begin{bmatrix} \alpha_1 & -(1-\alpha_1)\theta/2 & -(1-\alpha_1)(1-\theta)/2 \\ -(1-\alpha_2)/2 & \alpha_2 & 0 \\ -(1-\alpha_2)/2 & 0 & \alpha_2 \end{bmatrix}$$

The characteristic polynomial of J is

$$\begin{aligned} f(\lambda) = |\lambda \mathrm{I} - J| &= \begin{vmatrix} \lambda - \alpha_1 & (1-\alpha_1)\theta/2 & (1-\alpha_1)(1-\theta)/2 \\ (1-\alpha_2)/2 & \lambda - \alpha_2 & 0 \\ (1-\alpha_2)/2 & 0 & \lambda - \alpha_2 \end{vmatrix} \\ &= -(1-\alpha_2)/2 \begin{vmatrix} (1-\alpha_1)\theta/2 & (1-\alpha_1)(1-\theta)/2 \\ 0 & \lambda - \alpha_2 \end{vmatrix} + (\lambda - \alpha_2) \begin{vmatrix} \lambda - \alpha_1 & (1-\alpha_1)(1-\theta)/2 \\ (1-\alpha_2)/2 & \lambda - \alpha_2 \end{vmatrix}. \\ &= (\lambda - \alpha_2)((\lambda - \alpha_1)(\lambda - \alpha_2) - (1-\alpha_1)(1-\alpha_2)/4) \end{aligned}$$

Thereby one characteristic root of $f(\lambda)$ is $\lambda_1 = \alpha_2 \in (0, 1)$. To solve the other two characteristic roots of $f(\lambda)$, we consider the equation

$$(\lambda - \alpha_1)(\lambda - \alpha_2) - (1-\alpha_1)(1-\alpha_2)/4 = 0$$

or

$$4\lambda^2 - 4(\alpha_1 + \alpha_2)\lambda + \alpha_1\alpha_2 - (1-\alpha_1)(1-\alpha_2)/4 = 0$$

The other two characteristic roots of $f(\lambda)$ are

$$\lambda_\pm = (\alpha_1 + \alpha_2 \pm \sqrt{(\alpha_1+\alpha_2)^2 - 4\alpha_1\alpha_2 + (1-\alpha_1)(1-\alpha_2)})/2$$

The locally asymptotic stability condition for Bayesian Nash equilibrium is $1 \succ \lambda_+ \succ \lambda_- \succ -1$.
$\lambda_+ \prec 1$ is equivalent to

$$\sqrt{(\alpha_1+\alpha_2)^2 - 4\alpha_1\alpha_2 + (1-\alpha_1)(1-\alpha_2)} \prec 2 - (\alpha_1 + \alpha_2)$$

Square ends of the formula and have the following inequality

$$(1-\alpha_1)(1-\alpha_2)/4 \prec (1-\alpha_1)(1-\alpha_2)$$

This always holds for $\alpha_i \in (0, 1), \quad i = 1, 2.$
$\lambda_- \succ -1$ is equivalent to

$$2 + \alpha_1 + \alpha_2 \succ \sqrt{(\alpha_1+\alpha_2)^2 - 4\alpha_1\alpha_2 + (1-\alpha_1)(1-\alpha_2)}$$

Square ends of the formula

$$4 + 4(\alpha_1 + \alpha_2) \succ (1-\alpha_1)(1-\alpha_2)$$

This always holds for $\alpha_i \in (0, 1), \quad i = 1, 2.$

Let u_1, u_2, u_3 is the eigenvector of λ_1, λ_+, λ_- respectively. Since λ_1, λ_+, λ_- are different, u_1, u_2, u_3 are linearly independent. Any initial value X(0) can be represented as $X(0)=k_1u_1+k_2u_2+k_3u_3$. Therefore we have

$$X(t+1)=J^{k+1}X(0)+(\sum_{t=0}^{k}J^t)b=k_1\lambda^{k+1}u_1+k_2\lambda_+^{k+1}u_2+k_3\lambda_-^{k+1}u_3+(\sum_{t=0}^{k}J^t)b,\ \ (J^0=I)$$

When $t\to\infty$, X(t) converges to $(I-J)^{-1}b$, where

$$(I-J)^{-1}=\frac{1}{3}\begin{bmatrix}4/(1-\alpha_1) & -2\theta/(1-\alpha_2) & -2(1-\theta)/(1-\alpha_2)\\ -2/(1-\alpha_1) & (3+\theta)/(1-\alpha_2) & (1-\theta)/(1-\alpha_2)\\ -2/(1-\alpha_1) & \theta/(1-\alpha_2) & (4-\theta)/(1-\alpha_2)\end{bmatrix}$$

$$(I-J)^{-1}b=(q_1{}^*,q_2{}^*(c_h),q_2{}^*(c_l)).\quad\square$$

In this section, the result illustrates that under the assumption of two players' adaptive expectation, the Bayesian Nash equilibrium is always globally asymptotically stable. It is a conclusion different from that of the existing references. In previous literatures, the equilibrium is locally asymptotically stable only when parameters satisfy certain conditions.

4. The stability of Bayesian Nash equilibrium of dynamic Cournot duopoly model with heterogeneous expectations and asymmetric information

In most situations, firms use heterogeneous expectations more frequently. It is assumed that firm 1 has gradient rule based on marginal profit in Section 4, i. e., he adjusts output of next period according to his output of current period and estimation of marginal profit, while firm 2 uses adaptive expectation rule. The dynamic adjustment equations are:

$$\begin{cases}q_1{}'=q_1+vq_1(a-c_1-\theta q_{2h}-(1-\theta)q_{2l}-2q_1)\\ q_{2h}{}'=\alpha q_{2h}+(1-\alpha)(a-c_{2h}-q_1)/2\\ q_{2l}{}'=\alpha q_{2l}+(1-\alpha)(a-c_{2l}-q_1)/2\end{cases}\tag{10}$$

Where $v\succ 0$ represents firm 1's speed of output adjustment.

Fixed points of formula (10) are the solutions of algebraic equations as follows.

$$\begin{cases}q_1(a-c_1-\theta q_{2h}-(1-\theta)q_{2l}-2q_1)=0\\ (a-c_{2h}-q_1)/2-q_{2h}=0\\ (a-c_{2l}-q_1)/2-q_{2l}=0\end{cases}$$

The two fixed points are boundary equilibrium $E_0=(0,(a-c_{2h})/2,(a-c_{2l})/2)$ and Bayesian Nash equilibrium $E_*=(q_1{}^*,q_2{}^*(c_h),q_2{}^*(c_l))$ which is given by formula (7) and (8). The Jacobian matrix of formula (10) is

$$J=\begin{bmatrix}1+v(a-c_1-\theta q_{2h}-(1-\theta)q_{2l}-4q_1) & -vq_1\theta & -vq_1(1-\theta)\\ -(1-\alpha)/2 & \alpha & 0\\ -(1-\alpha)/2 & 0 & \alpha\end{bmatrix}$$

At boundary equilibrium E_0, the Jacobian matrix is

$$J_0=\begin{bmatrix}1+\frac{1}{2}v(a-c_1-(1-2\theta)\varepsilon) & 0 & 0\\ -(1-\alpha)/2 & \alpha & 0\\ -(1-\alpha)/2 & 0 & \alpha\end{bmatrix}$$

Under the assumption of formula (1), one characteristic root of J_0 is $\lambda_1=1+\frac{1}{2}v(a-c_1-(1-2\theta)\varepsilon)\succ 1$, the other two are $\lambda_2=\lambda_3=\alpha\in(0,1)$. Thus we have the following proposition.

Proposition 4. *If firm 1 chooses an adjustment rule based on marginal profit and firm 2 adopts an adaptive expectation rule, then the boundary equilibrium $E_0=(0,(a-c_{2h})/2,(a-c_{2l})/2)$ is a saddle point.*

At Bayesian Nash equilibrium E_*, the Jacobian matrix is

$$J_*=\begin{bmatrix}1-2vq_1 & -\theta vq_1 & -(1-\theta)vq_1\\ -(1-\alpha)/2 & \alpha & 0\\ -(1-\alpha)/2 & 0 & \alpha\end{bmatrix},$$

where $q_1=q_1{}^*$.

The characteristic polynomial of J^* is

$$f(\lambda)=|\lambda I-J_*|=\begin{vmatrix}\lambda+2vq_1-1 & \theta vq_1 & (1-\theta)vq_1\\ (1-\alpha)/2 & \lambda-\alpha & 0\\ (1-\alpha)/2 & 0 & \lambda-\alpha\end{vmatrix}$$

$$=-\frac{1-\alpha}{2}\begin{vmatrix}\theta vq_1 & (1-\theta)vq_1\\ 0 & \lambda-\alpha\end{vmatrix}+(\lambda-\alpha)\begin{vmatrix}\lambda+2vq_1-1 & (1-\theta)vq_1\\ (1-\alpha)/2 & \lambda-\alpha\end{vmatrix}$$

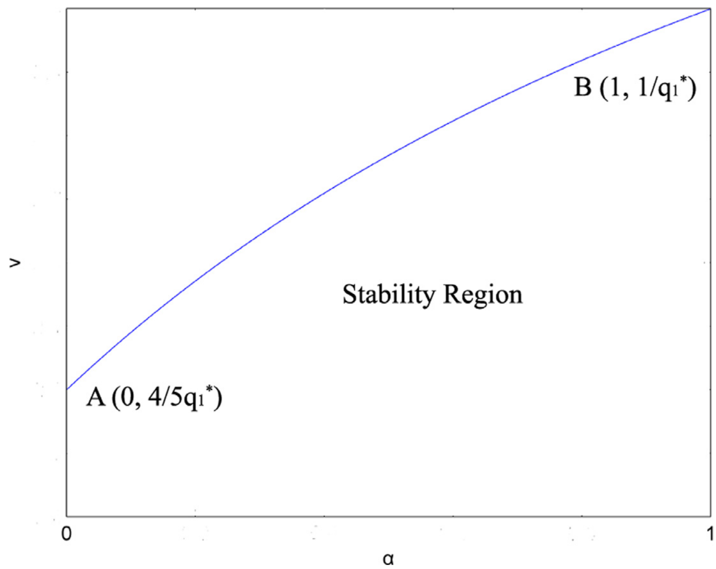

Fig. 1. The stability region of Bayesian Nash equilibrium.

$$= (\lambda-\alpha)((\lambda+2vq_1-1)(\lambda-\alpha)-(1-\alpha)(1-\theta)vq_1/2-(1-\alpha)\theta vq_1/2)$$

Therefore $f(\lambda)$ has one characteristic root $\lambda_1=\alpha\in(0,1)$, the other two characteristic roots satisfy

$$\lambda^2+(2vq_1-1-\alpha)\lambda-((2vq_1-1)\alpha+(1-\alpha)vq_1/2)=0$$

The other two characteristic roots are

$$\lambda_{\pm}=(1+\alpha-2vq_1\pm\sqrt{(1+\alpha-2vq_1)^2+2(3\alpha+1)vq_1-4\alpha})/2$$

The locally asymptotic stability condition for Bayesian Nash equilibrium E_* is $1\succ\lambda_+\succ\lambda_-\succ-1$, where $\lambda_+\prec1$ is equivalent to

$$\sqrt{(1+\alpha-2vq_1)^2+2(3\alpha+1)vq_1-4\alpha}\prec2-(1+\alpha-2vq_1)$$

Square ends of the formula and have the following inequality:

$$2(3\alpha+1)\prec8$$

From $\alpha\in(0,1)$, $\lambda_+\prec1$ holds.
$\lambda_-\succ-1$ is equivalent to

$$2+(1+\alpha-2vq_1)\succ\sqrt{(1+\alpha-2vq_1)^2+2(3\alpha+1)vq_1-4\alpha}$$

Square ends of the formula and have the following inequality:

$$4+4(1+\alpha-2vq_1)\succ2(3\alpha+1)vq_1-4\alpha$$

Therefore $v\prec4(1+\alpha)/(5+3\alpha)q_1$ holds. So we get Proposition 5.

Proposition 5. *If firm 1 chooses a gradient expectation rule and firm 2 adopts an adaptive expectation behavior, when* $v\prec4(1+\alpha)/(5+3\alpha)q_1^*$, *Bayesian Nash equilibrium is locally asymptotically stable.*

From Proposition 5, we obtain the stability region S for Bayesian Nash equilibrium, that is

$$S=\left\{\alpha\in(0,1),\ 0\prec v\prec V^{GA}\right\},$$

where the threshold is

$$V^{GA}(\alpha)=4(1+\alpha)/(5+3\alpha)q_1{}^* \tag{11}$$

It is also the boundary curve of stability region S. The intersection point of the boundary curve and $\alpha=0$ is $A(0,\ 4/5q_1{}^*)$. The intersection point of the boundary curve and $\alpha=1$ is $B(1,\ 1/q_1{}^*)$. From $\partial V^{GA}/\partial\alpha\succ0, \partial^2V^{GA}/\partial\alpha^2\prec0$, $V^{GA}(\alpha)$which is shown as Fig. 1, is a strictly increasing and concave curve.

When$(\alpha,\ v)\in$S(Stability Region), Bayesian Nash equilibrium is locally asymptotically stable, while if $(\alpha,v)\bar{\in}S$, a bifurcation even chaos phenomenon could be seen by numerical simulation.

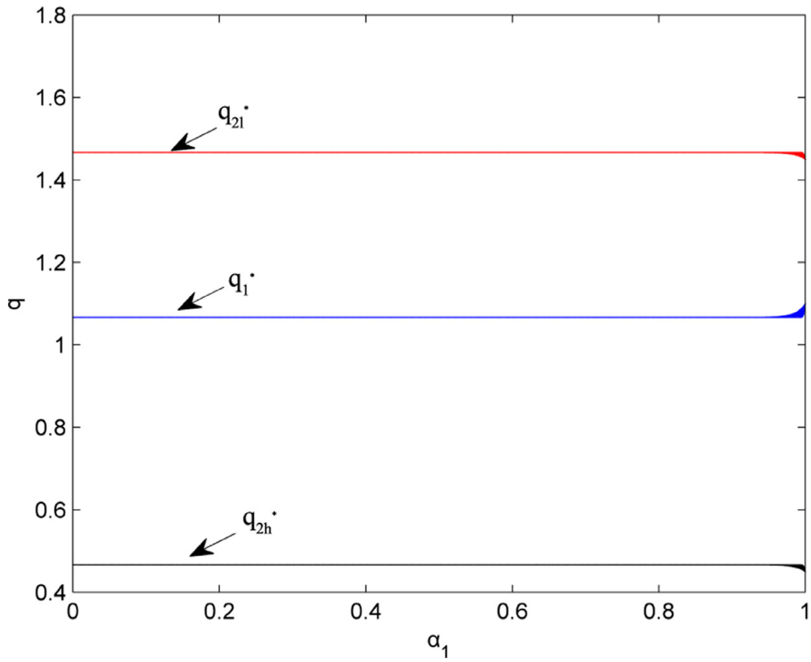

Fig. 2. Bayesian Nash equilibrium with respect to α_1.

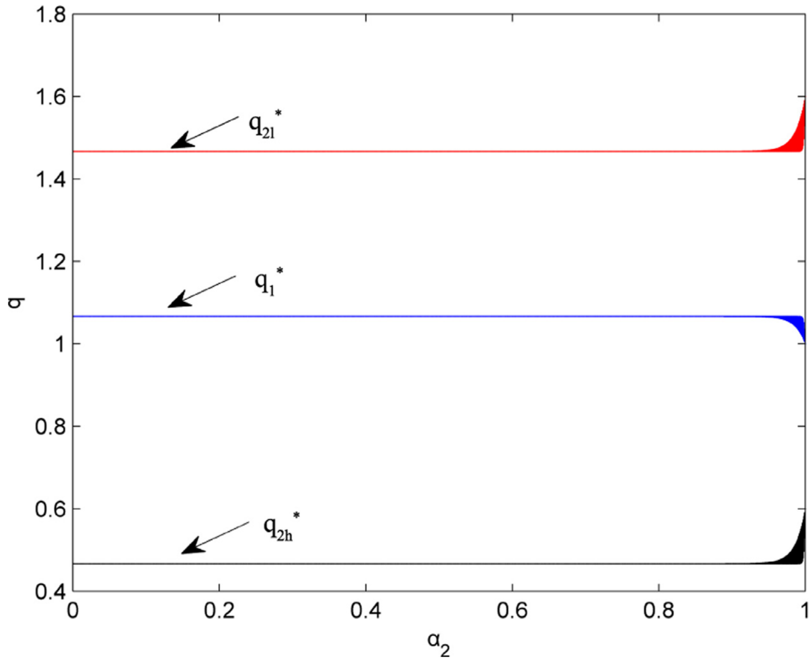

Fig. 3. Bayesian Nash equilibrium with respect to α_2.

We substitute $q_1{}^* = (a - c_1 + (2\theta - 1)\varepsilon)/3$ into formula (11) and get the following equation:

$$V^{GA}(\alpha)=12(1+\alpha)/(5+3\alpha)(a-c_1+(2\theta-1)\varepsilon)$$

From this expression, we obtain Proposition 6.

Proposition 6. *Consider two heterogeneous firms. It is assumed that firm 1 chooses a gradient expectation rule and firm 2 adopts an adaptive expectation behavior, the stability of Bayesian Nash equilibrium increases as the parameter α or c_1 increases, and it*

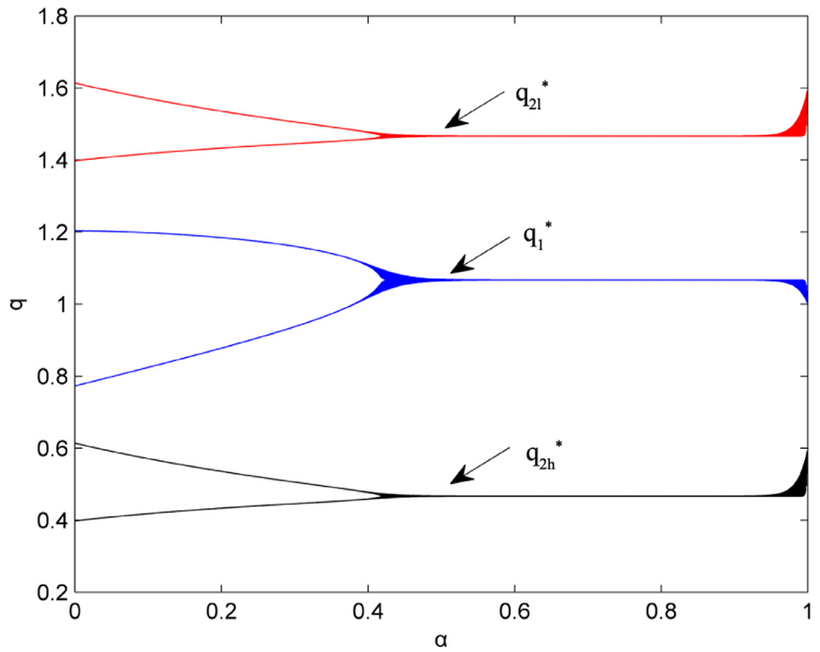

Fig. 4. Bifurcation diagram with respect to α.

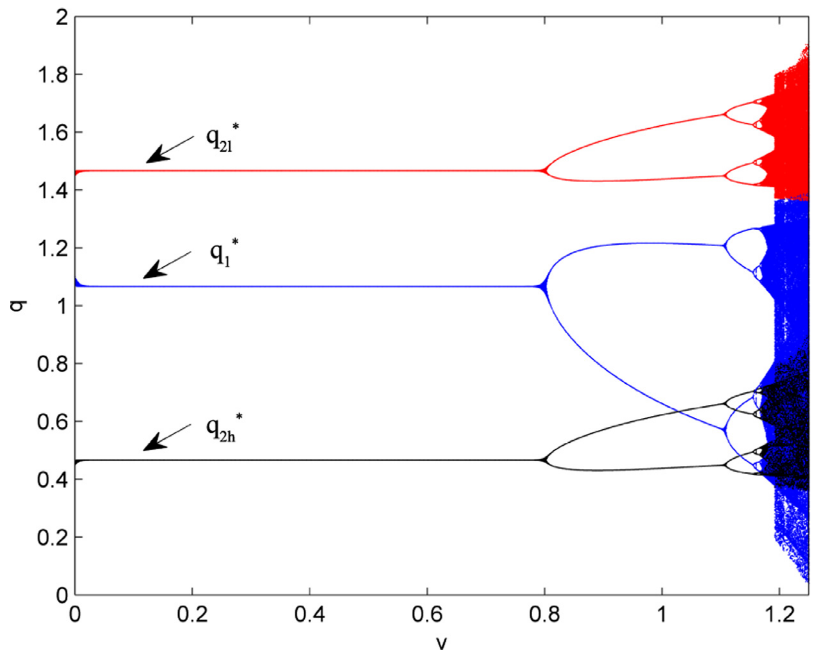

Fig. 5. Bifurcation diagram with respect to v.

decreases as the parameter a or θ *increases. When* $\theta \succ \frac{1}{2}$*, the stability decreases as the parameter* ε *increases. When* $\theta \prec \frac{1}{2}$*, the stability increases as the parameter* ε *increases.*

5. Simulation

In this section, lots of behaviors of dynamic Cournot duopoly model with asymmetric information are presented by simulation. A series of bifurcation and chaos with respect to α_1, α_2, v and θ, largest Lyapunov exponent, strange attractors, and sensitivity on initial values are shown in figures. Firstly, we consider the stability of equilibrium with asymmetric informa-

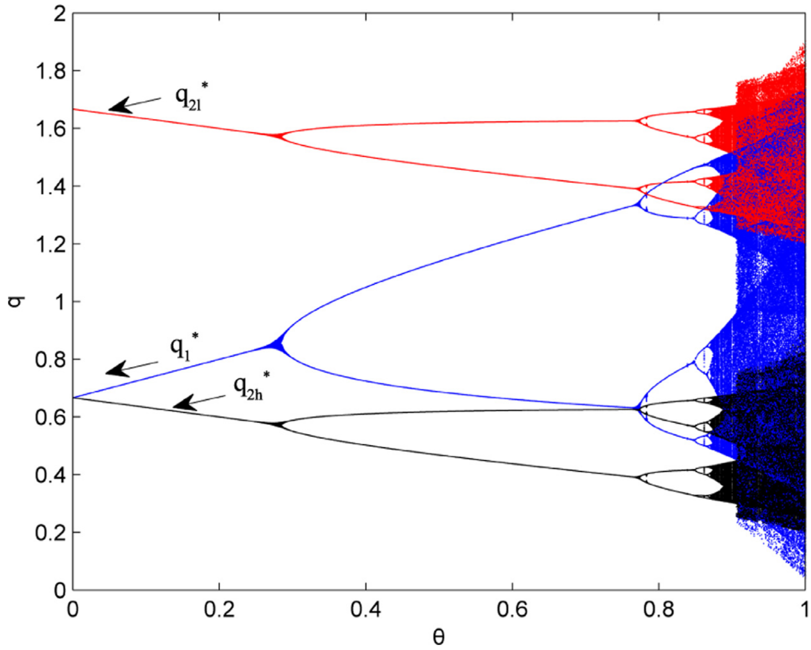

Fig. 6. Bifurcation diagram with respect to θ.

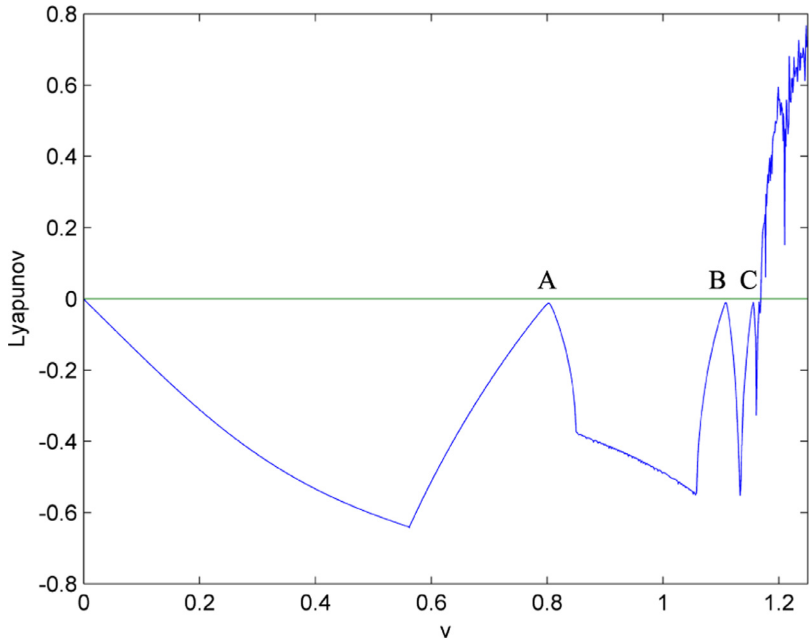

Fig. 7. Lyapunov exponent diagram with respect to v.

tion and two players with adaptive expectation in Section 4. We set parameters as follows: $a=5$, $c_1=2$, $\varepsilon=1$, $\theta=0.6$, thus $c_h=3$, $c_l=1$, $q_1{}^*=1.0667$, $q_{2h}{}^*=0.4667$, $q_{2l}{}^*=1.4667$.

From formula (9), let $\alpha_2=0.6$, we get Bayesian Nash equilibrium with respect to α_1 as Fig. 2 shows. Let $\alpha_1=0.5$, we get Bayesian Nash equilibrium with respect to α_2 as Fig. 3 shows. Although α_1 and α_2 are two different parameters, Bayesian Nash equilibrium curves in Figs. 2 and 3 are almost the same, which verifies Proposition 3: consider two players with adaptive expectation, Bayesian Nash equilibrium of dynamic Cournot duopoly model with asymmetric information is always globally asymptotically stable.

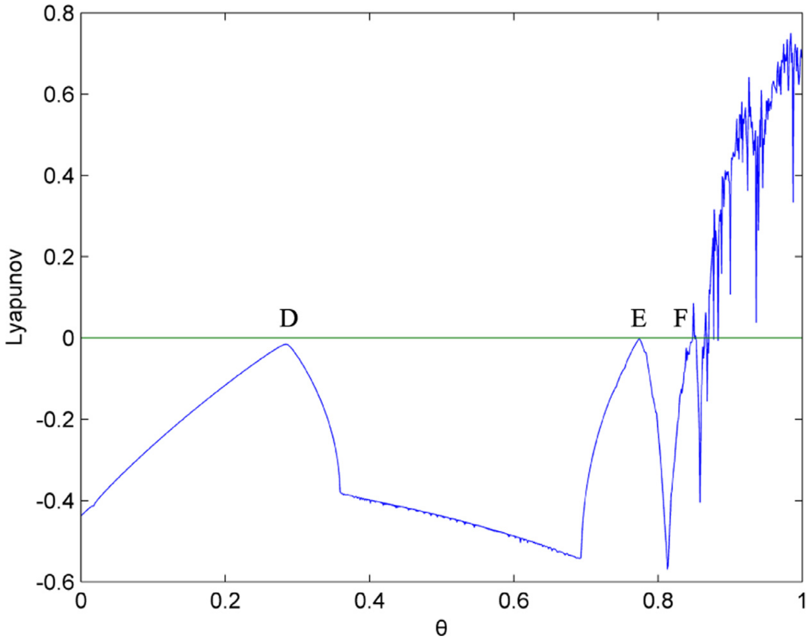

Fig. 8. Lyapunov exponent diagram with respect to θ.

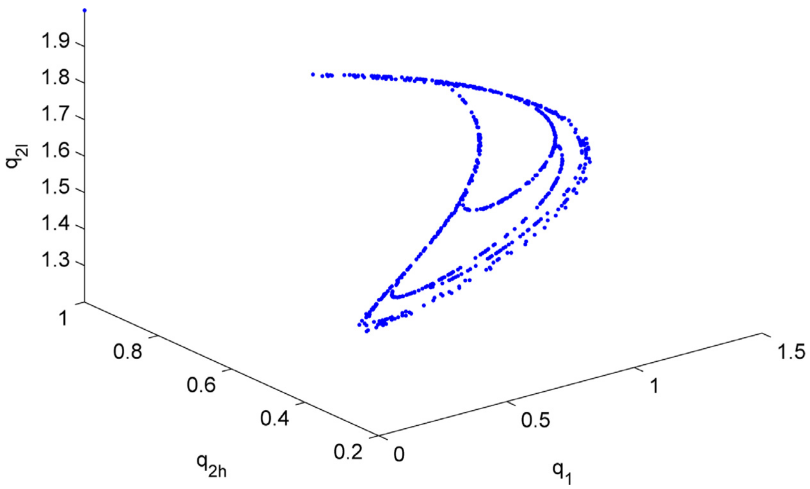

Fig. 9. Strange attractor for $v = 1.22$.

Next we consider the stability of equilibrium with two heterogeneous expectations players and asymmetric information in Section 5. We set parameters as follows: $a = 5$, $c_1 = 2$, $\varepsilon = 1$, $\theta = 0.6$, thus $c_h = 3$, $c_l = 1$, ${q_1}^* = 1.0667$, ${q_{2h}}^* = 0.4667$, ${q_{2l}}^* = 1.4667$.

From formula (10), let $v = 0.85$, we get bifurcation diagram with respect to α as Fig. 4 shows; let $\alpha = 0.2$, we get bifurcation diagram with respect to v as Fig. 5 shows. From the stability region for Bayesian Nash equilibrium in Fig. 1, $(\alpha, v) = (0.4167, 0.85)$ in Fig. 1 corresponds to the bifurcation point $\alpha = 0.4167$ in Fig. 4. $(\alpha, v) = (0.2, 0.8035)$ in Fig. 1 corresponds to the bifurcation point $v = 0.8035$ in Fig. 5. The outputs of firms have the experience from bifurcation to equilibrium as α increases, and it has the experience from equilibrium to bifurcation and even chaos as v increases. Adaptive expectation is such a convex combination of firm 2's period output and reply function. The bigger α is, the more attention he has to pay period output and the less attention he has to pay reply function. v is firm 2's adjustment speed of output. The bigger v is, the easier chaos could occur.

Fix $a = 5$, $c_1 = 2$, $\varepsilon = 1$, let $\alpha = 0.2$, $v = 1$, we get bifurcation diagram with respect to θ as Fig. 6 shows. The outputs of firms have the experience from equilibrium to bifurcation and even chaos as θ increases. θ is a probability parameter which describes the uncertainty cost function. Large probability of high-cost production yields easier chaos in duopoly market. Furthermore, from Proposition 2, in formula (7) and (8), ${q_1}^* = (a - c_1 + (2\theta - 1)\varepsilon)/3, {q_2}^*(c_2) =$

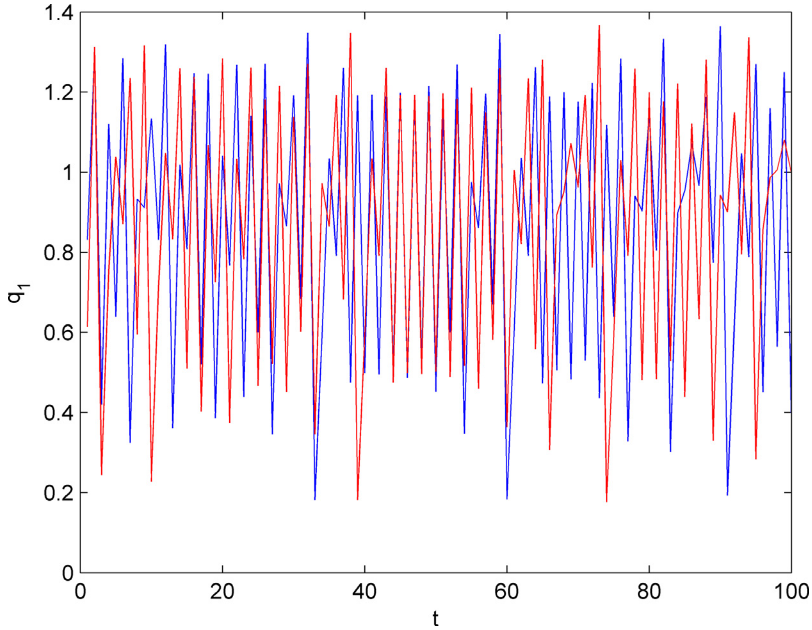

Fig. 10. Two orbits of q_1 with initial values for $v = 1.22$.
Notes: The blue curve represents a group of initial values (q_{10}, q_{2h0}, q_{2l0}) = (1.1, 0.6, 1.6). The red curve represents a group of initial values (q_{10}, q_{2h0}, q_{2l0}) = (1.101, 0.6, 1.6). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

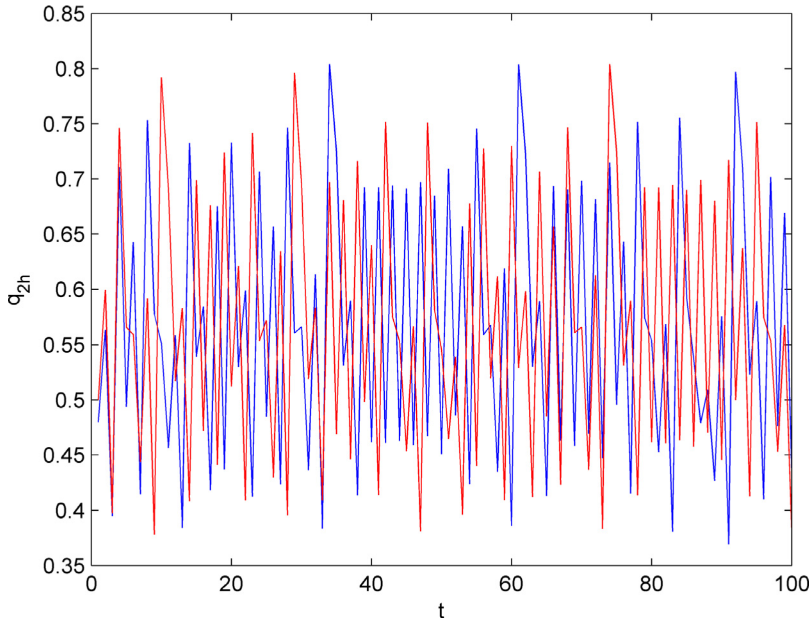

Fig. 11. Two orbits of q_{2h} with initial values for $v = 1.22$.
Notes: The blue curve represents a group of initial values (q_{10}, q_{2h0}, q_{2l0}) = (1.1, 0.6, 1.6). The red curve represents a group of initial values (q_{10}, q_{2h0}, q_{2l0}) = (1.1, 0.601, 1.6). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

$\begin{cases} (a - c_1 - (1+\theta)\varepsilon)/3 & c_2 = c_h \\ (a - c_1 + (2-\theta)\varepsilon)/3 & c_2 = c_l \end{cases}$. ${q_1}^*$ increases and ${q_{2h}}^*$, ${q_{2l}}^*$ decrease as the parameter θ increases, which is shown in Fig. 6.

Figs. 7 and 8 are the largest Lyapunov exponents with respect to v and θ. The first, second and third bifurcation point in Fig. 5 corresponds to A (0.8030, −0.0125), B (1.1090, −0.0104) and C (1.1560, −0.0083) in Fig. 7, respectively. The first, second and third bifurcation point in Fig. 6 corresponds to D (0.2890, −0.0199), E (0.7740, −0.0020) and F (0.8450, −0.0100) in Fig. 8, respectively. When the largest Lyapunov exponent is larger than zero, chaos occurs.

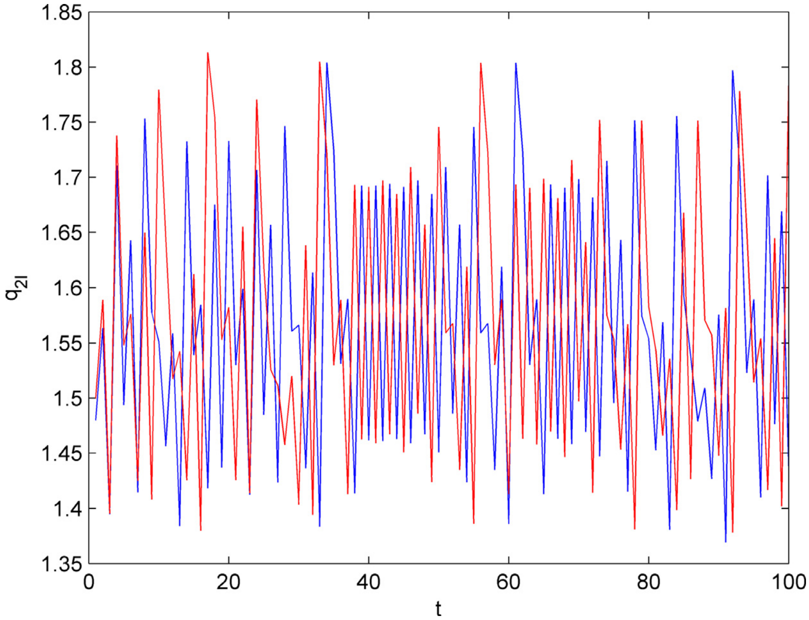

Fig. 12. Two orbits of q_{2l} with initial values for $v = 1.22$.
Notes: The blue curve represents a group of initial values (q_{10}, q_{2h0}, q_{2l0}) = (1.1, 0.6, 1.6). The red curve represents a group of initial values (q_{10}, q_{2h0}, q_{2l0}) = (1.1, 0.6, 1.601). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

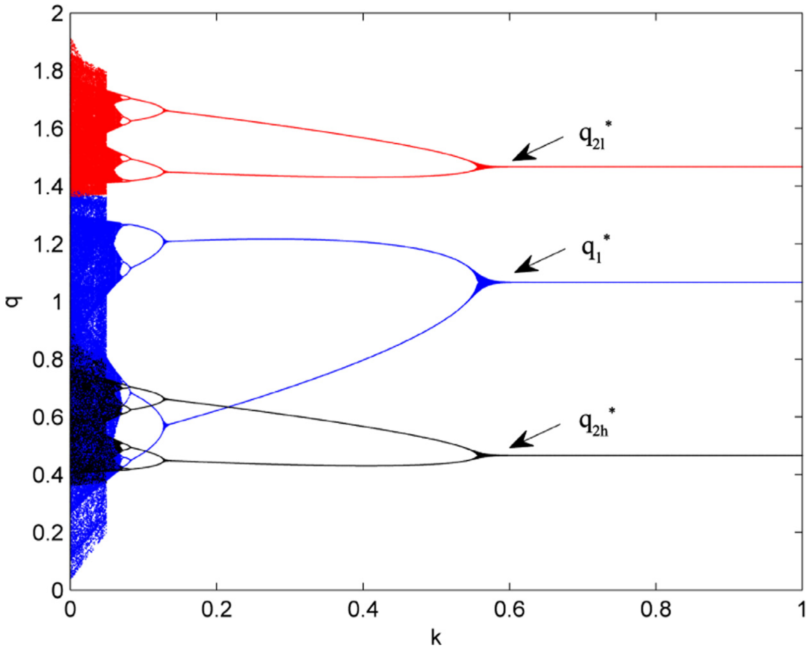

Fig. 13. Bifurcation diagram with respect to control factor k.

Fig. 9 shows strange attractors for a bigger v. It implies the system experiences chaos when $v = 1.22$.

We also need to discuss the sensitivity on initial values. From Figs. 10–12, $v = 1.22$ is fixed.

As t increases, a tiny change $\Delta = 0.001$ of q_1, q_{2h}, q_{2l} could cause a dramatic fluctuation when the system is in chaos ($v = 1.22$) from Figs. 10–12.

Fig. 13 shows the map of bifurcation with respect to control factor k. Feedback control method is used to delay and eliminate chaos. The outputs of manufactures finally stabilize at the equilibrium levels $q_1{}^* = 1.0667$, $q_{2h}{}^* = 0.4667$, $q_{2l}{}^* = 1.4667$.

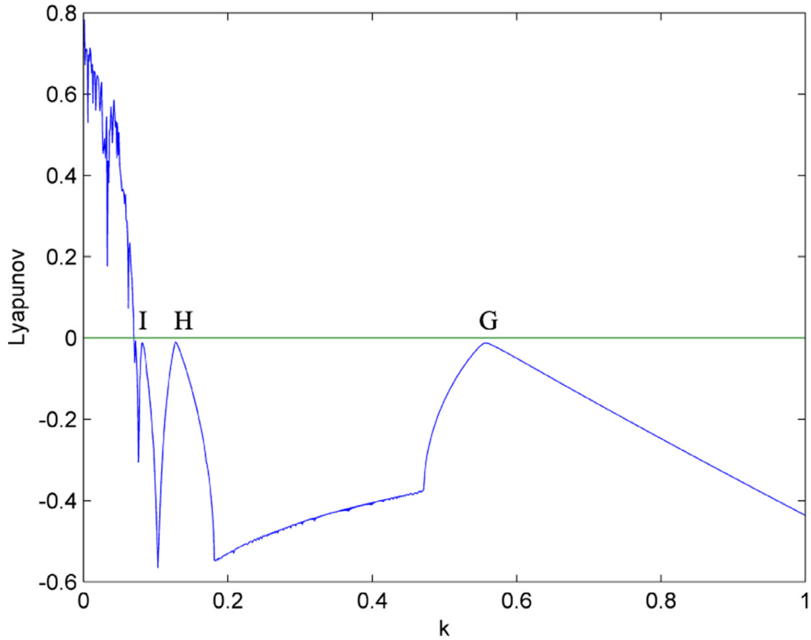

Fig. 14. Lyapunov exponent diagram with respect to k.

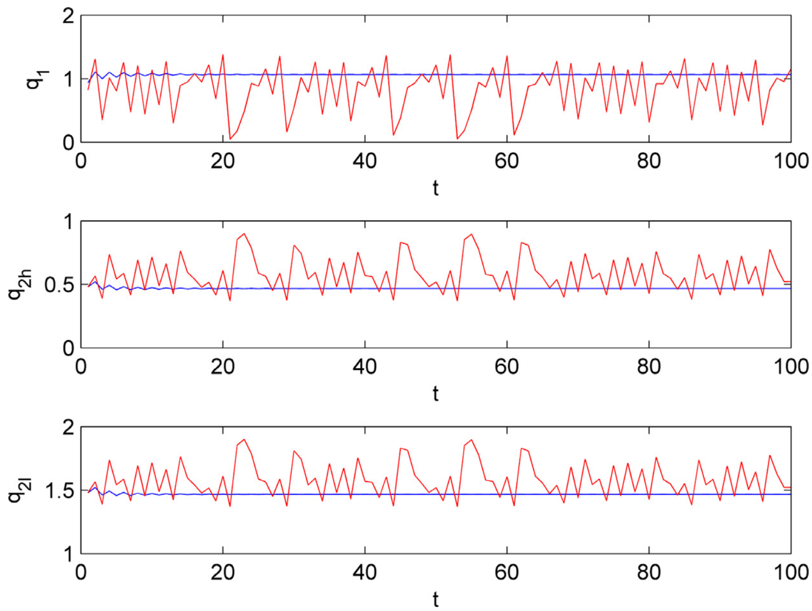

Fig. 15. Effects of control factor k on q_1, q_{2h}, q_{2l}.
Notes: The red curve in three subgraphs represents the chaotic status of q_1, q_{2h}, q_{2l} respectively. The blue curve in three subgraphs represents the Bayesian Nash equilibrium output of q_1, q_{2h}, q_{2l} respectively with a control factor k ($k = 0.7$). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

The largest Lyapunov exponent with respect to k is drawn in Fig. 14. The first, second and third bifurcation point in Fig. 13 corresponds to G (0.5690, −0.0202), H (0.1270, −0.0109) and I (0.0810, −0.0117) in Fig. 14, respectively. When the largest Lyapunov exponent is smaller than zero, equilibrium occurs.

Figs. 15 and 16 represent the effects of control factor k on outputs and profits. The outputs and profits experience chaos and bifurcation, maintain equilibrium finally ($q_1^* = 1.0667$, $q_{2h}^* = 0.4667$, $q_{2l}^* = 1.4667$; $\pi_1^* = 1.7778$, $\pi_{2h}^* = 0.2178$, $\pi_{2l}^* = 2.1511$).

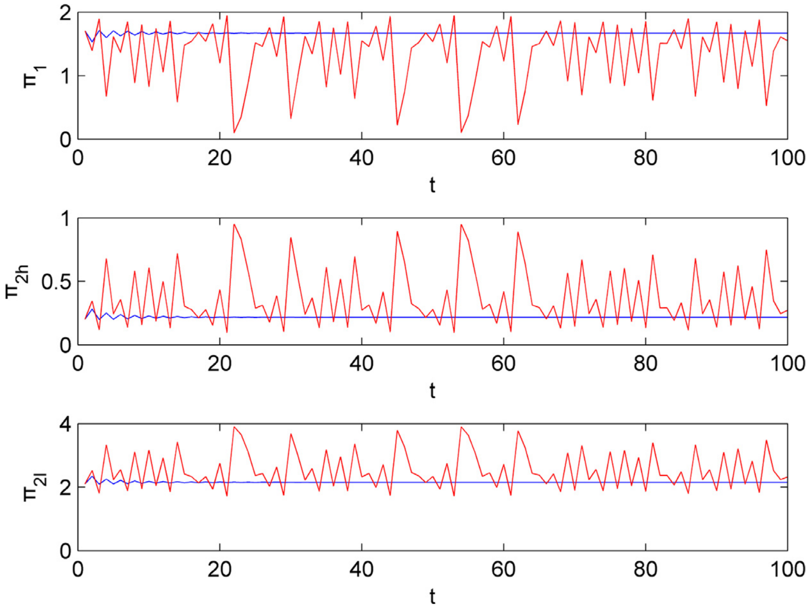

Fig. 16. Effects of control factor k on π_1, π_{2h}, π_{2L}.
Notes: The red curve in three subgraphs represents the chaotic status of π_1, π_{2h}, π_{2l} respectively. The blue curve in three subgraphs represents the Bayesian Nash equilibrium profit of q_1, q_{2h}, q_{2l} respectively with a control factor k ($k = 0.7$). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

6. Conclusion

In similar research articles, although considering bounded rationality and asymmetric information, whether Bayesian Nash equilibrium could be achieved by dynamic system is not demonstrated. There are few literatures analyzing complex dynamics behavior of game of incomplete information.

In this paper, we construct a dynamic Cournot duopoly model with asymmetric information. Specially, we have two assumptions. Firstly, firm 2 has a marginal cost which is his private information. Secondly, firm 1's marginal cost is a common knowledge. We use the above assumptions to introduce asymmetry information of firms and describe actual economic problems. In our models, based on adaptive expectation and bounded rationality, the probability distribution parameter θ (firm 2's probability of high cost) plays an important role in chaos occurrence.

In theoretical analysis of Sections 4 and 5, we give two behavioral rules: adaptive expectation and an adjustment rule based on marginal profit. Globally and locally asymptotically stability of Bayesian Nash equilibrium is proved in Propositions 3 and 5 respectively. They illustrate that different expectations have effect on stability of Bayesian Nash equilibrium.

In numerical simulation section, we observed abundant and complex phenomena which verify the theoretical results. No matter how expectation parameters α_1 or α_2 changes, equilibrium would remain. But with parameter θ or v (firm 1's adjustment speed of output) increasing, bifurcation or even chaos, is presented in simulation. The influence of parameters on the stability of system is further analyzed. The above results imply that once a firm chooses a production form (high cost or low cost), the effect of an adjustment rule based on marginal profit on market stability is greater than expectation adjustment behavior. On the contrary, if expectation and adjustment speed of output parameters are determined, a high-cost firm is more likely to cause chaos in duopoly market. In addition, a three-dimensional strange attractor region is presented. Sensitivity on initial values is demonstrated. Chaos is controlled by delay feedback control method.

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